# PRINCIPLES OF MATHEMATICAL ANALYSIS 

LIAM KOUCH

## Contents

1. The Real and Complex Number System ..... 2
2. Basic Topology ..... 7
3. Numerical Sequences and Series ..... 22
4. Continuity ..... 25

## 1. The Real and Complex Number System

Theorem (1.20). Part (a) If $x, y \in \mathbb{R}$ and $x>0$, then there is a positive integer $n$ such that $n x>y$. Part (b) If $x, y \in \mathbb{R}$ and $x<y$ then there exists a $p \in \mathbb{Q}$ such that $x<p<y$.

Proof. We prove part (a) first. Consider the set $A=\{n x \mid n \in \mathbb{N}\}$. For the sake of contradiction suppose there does not exist a natural number $n$ such that $n x>y$. Then for all $a \in A$ we have $a \leq y$. Then $A$ is bounded above so then let $\alpha=\sup A$. Since $0>-x$ and $\alpha>0$ it follows $\alpha>\alpha-x$. Then $\alpha-x$ is not an upper bound. If $\alpha-x$ is not an upper bound then there exists some natural number $m$ such that $m x>\alpha-x$. Then $(m+1) x>\alpha$. This is a contradiction. Hence, there exists a natural number $n$ such that $n x>y$.

Now we prove part (b). Let $x, y \in \mathbb{R}$ and $x<y$. If $x \leq 0$ then by the archimedean property (part a) there exists a natural number $k$ such that $k>-x$. It follows $x+k>0$. Then we have

$$
0<x+k<y+k
$$

If $x>0$ then let $k=0$. Let $x_{0}=x$ and $y_{0}=y$. Then we have $0<x_{0}<y_{0}$. Now we show there exists a rational number $p$ such that $x_{0}<p<y_{0}$.

Since $y_{0}>x_{0}$ then $y_{0}-x_{0}>0$. By the archimedean property there exists a natural number $m$ such that $m\left(y_{0}-x_{0}\right)>1$. Then we have $m y_{0}-m x_{0}>1$ so then

$$
m x_{0}<m x_{0}+1<m y_{0}
$$

Now we show there exists a natural number $d$ such that $m x_{0}<d \leq m x_{0}+1$. Consider the set $B=\left\{n \in \mathbb{N} \mid n>m x_{0}\right\}$. Note $B \subset \mathbb{N}$ so then there is a least element in $B$. Take $n_{0}$ to be the least element. Then $n_{0} \leq j$ for all $j \in B$. Now we show $n_{0} \leq m x_{0}+1$. For the sake of contradiction, suppose $n_{0}>m x_{0}+1$. Then $n_{0}-1>m x_{0}$. Moreover, since $m x_{0} \geq 1$ and $n_{0}>m x_{0}$ it follows $n_{0}>1$ so then $n_{0}-1 \geq 1$ so $n_{0}-1$ is also a natural number. So then $n_{0}-1 \in \mathbb{N}$ and $n_{0}-1>m x_{0}$ so $n_{0}-1 \in B$. This is a contradiction since $n_{0}$ is the least element in $B$. It follows $n_{0} \leq m x_{0}+1$.

Altogether we have $m x_{0}<n_{0} \leq m x_{0}+1$. Then

$$
m x_{0}<n_{0} \leq m x_{0}+1<m y_{0}
$$

and hence $m x_{0}<n_{0}<m y_{0}$ Since $m \in \mathbb{N}$ there exists $1 / m$. Then

$$
x_{0}<\frac{n_{0}}{m}<y_{0}
$$

Recall $x_{0}=x+k$ and $y_{0}=y+k$. Then we have

$$
\begin{aligned}
& x<\frac{n_{0}}{m}+k<y \\
& x<\frac{n_{0}+m k}{m}<y
\end{aligned}
$$

Since $n_{0}, m \in N$ and $k \in \mathbb{Z}$ it follows $\left(n_{0}+m k\right) / m \in \mathbb{Q}$. This concludes the proof.
Exercise. (8) Prove that no order can be defined in the complex field that turns it into an ordered field.

Proof. In order for the complex field to be an ordered field, the set of complex numbers must be an ordered set. This means by definition of an ordered set that for any $x, y \in \mathbb{C}$ we must have exactly one of the following to be true

$$
x<y, x=y, \text { or } x>y
$$

Then this should be true for $i$ and 0 . For the sake of contradiction suppose one of these is exactly true for $i$ and 0 . Then there are three cases to test. First, suppose $i=0$. Since 0 is the additive identity it follows we must have $i+z=z$ for all $z \in \mathbb{C}$ which is false. To be more explicit let the complex numbers be a set of two-tuples of real numbers. Then addition in $\mathbb{C}$ is defined as $(a, b)+_{\mathbb{C}}(c, d)=(a+b, c+d)$ and $i=(0,1)$ and zero formally is $0=(0,0)$. It follows $i \neq 0$ since $(0,1) \neq(0,0)$. Second, suppose $i>0$. Then by Definition $1.17(\mathrm{ii})$ of Rudin it follows

$$
i^{2}>0
$$

and by definition of $i$ we have

$$
-1>0
$$

Then applying Definition 1.17(ii) again, we get

$$
-1(i)>0
$$

By Proposition 1.16(c) it follows

$$
-(1 i)>0
$$

and by (M4) we have

$$
-i>0
$$

Then adding $i$ to both sides

$$
i+(-i)>i+0
$$

By (A5) (additive inverse) we have

$$
0>i+0
$$

By (A2) (commutativity) we have

$$
0>0+i
$$

By (A4) (additive identity) we have

$$
0>i
$$

But this is a contradiction. Third, suppose $i<0$. Then

$$
i+(-i)<0+(-i)
$$

By (A5) (addition inverse) we have

$$
0<0+(-i)
$$

By (A4) (addition identity) we have

$$
0<-i
$$

Since $-i>0$ by Definition 1.17(ii) we have

$$
(-i)(-i)>0
$$

By Proposition 1.16(d) we have

$$
i^{2}>0
$$

By definition of $i$ we have

$$
-1>0
$$

Since $-1>0$ and $-i>0$ by Definition 1.17 (ii) we have

$$
(-1)(-i)>0
$$

By Proposition 1.16(d) we have

$$
1 i>0
$$

By axiom (M4) we have

$$
i>0
$$

This is a contradiction. In all three cases we have reached contradictions. Therefore, the set of complex numbers cannot be an ordered field.

Exercise. (12) If $z_{1}, \ldots, z_{n}$ are complex prove that

$$
\left|z_{1}+z_{2}+\cdots+z_{n}\right| \leq\left|z_{1}\right|+\left|z_{2}\right|+\cdots+\left|z_{n}\right|
$$

Proof. First we prove that $\left|z_{1}+z_{2}\right| \leq\left|z_{1}\right|+\left|z_{2}\right|$. And before this is prove we prove $\operatorname{Re}(\lambda) \leq|\lambda|$ where $\lambda \in \mathbb{C}$. If $\lambda$ is a complex number it follows $\lambda=a+b i$ for some $a, b \in \mathbb{R}$. Then $\operatorname{Re}(\lambda)=a$. And $|\lambda|=\sqrt{a^{2}+b^{2}}$. Then we have

$$
\operatorname{Re}(\lambda)=a \leq \sqrt{a^{2}} \leq \sqrt{a^{2}+b^{2}}=|\lambda|
$$

It follows $\operatorname{Re}(\lambda) \leq|\lambda|$. Let $\lambda=z \bar{w}$ where $z, w \in \mathbb{C}$. It follows

$$
2 \operatorname{Re}(z \bar{w}) \leq 2|z \bar{w}|
$$

which is the same as

$$
z \bar{w}+\bar{z} w \leq \sqrt{z \overline{w z} w}
$$

Adding $z \bar{z}+w \bar{w}$ gives

$$
z \bar{z}+z \bar{w}+\bar{z} w+w \bar{w} \leq z \bar{z}+\sqrt{z \overline{w z} w}+w \bar{w}
$$

This is equivalent to

$$
(z+w)(\bar{z}+\bar{w}) \leq(\sqrt{z \bar{z}}+\sqrt{w \bar{w}})^{2}
$$

This is the same as

$$
|z+w|^{2} \leq(|z|+|w|)^{2}
$$

Taking the square root of both sides gives

$$
|z+w| \leq|z|+|w|
$$

Thus, the triangle inequality holds for all complex numbers. Now we prove the main statement of this exercise by induction. Let $P(n)$ denote the truth of the statement

$$
\left|z_{1}+z_{2}+\cdots+z_{n}\right| \leq\left|z_{1}\right|+\left|z_{2}\right|+\cdots+\left|z_{n}\right|
$$

Thus we just proved $P(2)$ is true. And we know $P(1)$ is true since $|z| \leq|z|$ for any complex number. Moving on to the inductive step, we show that $P(n)$ implies $P(n+1)$. Suppose $P(n)$ is true and we have

$$
\left|z_{1}+z_{2}+\cdots+z_{n}+z_{n+1}\right|
$$

Then let $r=z_{n}+z_{n+1}$. Then we have

$$
\left|z_{1}+z_{2}+\cdots+z_{n-1}+r\right|
$$

Since we now have $n$ complex numbers and we know $P(n)$ is true we have the following inequality.

$$
\left|z_{1}+z_{2}+\cdots+z_{n}+z_{n+1}\right| \leq\left|z_{1}\right|+\left|z_{2}\right|+\cdots+\left|z_{n-1}\right|+\left|z_{n}+z_{n+1}\right|
$$

But we already know $P(2)$ is true so we have
$\left|z_{1}+z_{2}+\cdots+z_{n}+z_{n+1}\right| \leq\left|z_{1}\right|+\left|z_{2}\right|+\cdots+\left|z_{n-1}\right|+\left|z_{n}+z_{n+1}\right| \leq\left|z_{1}\right|+\left|z_{2}\right|+\cdots+\left|z_{n-1}\right|+\left|z_{n}\right|+\left|z_{n+1}\right|$
Thus $P(n+1)$ holds. This concludes the induction.

Exercise. (13) If $x, y$ are complex prove that

$$
\| x|-|y|| \leq|x-y|
$$

Proof. For any $a, b, c, d \in \mathbb{R}$ the following inequality holds.

$$
\begin{aligned}
(a d-b c)^{2} & \geq 0 \\
a^{2} d^{2}-2 a b c d+b^{2} c^{2} & \geq 0 \\
a^{2} c^{2}+a^{2} d^{2}+b^{2} c^{2}+b^{2} d^{2} & \geq a^{2} c^{2}+2 a b c d+b^{2} d^{2} \\
\left(a^{2}+b^{2}\right)\left(c^{2}+d^{2}\right) & \geq(a c+b d)^{2} \\
\sqrt{\left(a^{2}+b^{2}\right)\left(c^{2}+d^{2}\right)} & \geq a c+b d \\
-2 \sqrt{\left(a^{2}+b^{2}\right)\left(c^{2}+d^{2}\right)} & \leq-2(a c+b d) \\
\left(a^{2}+b^{2}\right)-2 \sqrt{\left(a^{2}+b^{2}\right)\left(c^{2}+d^{2}\right)}+\left(c^{2}+d^{2}\right) & \leq a^{2}-2 a c+c^{2}+b^{2}-2 b d+d^{2} \\
\left(\sqrt{a^{2}+b^{2}}-\sqrt{c^{2}+d^{2}}\right)^{2} & \leq(a-c)^{2}+(b-d)^{2} \\
\|a+b i|-| c+d i\|^{2} & \leq|(a-c)+(b-d) i|^{2} \\
\|a+b i|-| c+d i\| & \leq|(a+b i)-(c+d i)|
\end{aligned}
$$

Any pair of complex numbers $x, y \in \mathbb{C}$ can be written in the above form where $x=a+b i$ and $y=c+d i$. It follows the inequality holds.
Exercise. (17) Prove that

$$
|x+y|^{2}+|x-y|^{2}=2|x|^{2}+2|y|^{2}
$$

if $x \in \mathbb{R}^{k}$ and $y \in \mathbb{R}^{k}$.
Proof. The first and last equations are true by definition of the norm in euclidean k-space.

$$
\begin{aligned}
|x+y|^{2}+|x-y|^{2} & =\sum_{i=1}^{k}\left(x_{i}+y_{i}\right)^{2}+\sum_{i=1}^{k}\left(x_{i}-y_{i}\right)^{2} \\
& =\sum_{i=1}^{k}\left(x_{i}+y_{i}\right)^{2}+\left(x_{i}-y_{i}\right)^{2} \\
& =\sum_{i=1}^{k} x_{i}^{2}+2 x_{i} y_{i}+y_{i}^{2}+x_{i}^{2}-2 x_{i} y_{i}+y_{i}^{2} \\
& =\sum_{i=1}^{k} 2 x_{i}^{2}+2 y_{i}^{2} \\
& =2 \sum_{i=1}^{k} x_{i}^{2}+2 \sum_{i=1}^{j} y_{i}^{2} \\
& =2|x|^{2}+2|y|^{2}
\end{aligned}
$$

Exercise (14). If $z$ is a complex number such that $|z|=1$, compute

$$
\frac{|1+z|^{2}+|1-z|^{2}}{5}
$$

Proof.

$$
\begin{aligned}
|1+z|^{2}+|1-z|^{2} & =(1+z)(1+\overline{z d})+(1-z)(1-\overline{z d}) \\
& =1+z+\overline{z d}+z \overline{z d}+1-z-\overline{z d}+z \overline{z d} \\
& =2+2 z \overline{z d}
\end{aligned}
$$

## 2. Basic Topology

Exercise. (5) Construct a bounded set of real numbers with exactly three limit points.
Let the set be $\left\{\left.\frac{1}{n} \right\rvert\, n \in \mathbb{N}\right\} \cup\left\{\left.1+\frac{1}{n} \right\rvert\, n \in \mathbb{N}\right\} \cup\left\{\left.2+\frac{1}{n} \right\rvert\, n \in \mathbb{N}\right\}$
Exercise. (6) Let $E^{\prime}$ be the set of all limit points of a set $E$. Prove that $E^{\prime}$ is closed. Prove that $E$ and $\bar{E}$ has the same limit points. Do $E$ and $E^{\prime}$ always have the same limit points?
Proof. Suppose $E^{\prime}$ is the set of all limit points of $E$. We show that $E^{\prime}$ is closed. Suppose $p$ is a limit point of $E^{\prime}$. Then by definition of a limit point for all $\epsilon_{1}>0$ there exists $q \neq p$ such that $d(p, q)<\epsilon_{1}$ and $q \in E^{\prime}$. If $q$ in $E^{\prime}$ then $q$ must be a limit point of $E$. Then by definition of a limit point we have for all $\epsilon_{2}>0$ there exists $t \neq q$ such that $d(q, t)<\epsilon_{2}$ and $t \in E$. Moreover, we know there exists some $t \neq p$ since for any neighborhood $N_{\epsilon_{2}}(q)$ there must be an infinite number of points $t$ such that $d(q, t)<\epsilon_{2}$ and $t \neq q$. Otherwise, there would be a finite number and then there would be some closest element say $t_{0}$ which would imply the intersection of neighborhood $N_{d\left(q, t_{0}\right)}(q)$ with $E$ would be empty which implies $q$ would not be a limit point. Suppose we have some neighborhood $N_{\delta}(p)$ where $\delta>0$. Then we can choose $t \in E$ such that

$$
d(p, t) \leq d(p, q)+d(q, t)=\epsilon_{1}+\epsilon_{2}<\delta
$$

where $t \neq p$ since $\epsilon_{1}$ and $\epsilon_{2}$ can be arbitrarily small. It follows $p$ is a limit point of $E$. Since $E^{\prime}$ is the set of all limit points of $E^{\prime}$ it follows $p \in E^{\prime}$. But we assumed $p$ was a limit point of $E^{\prime}$. It follows $E^{\prime}$ is closed.
Exercise. (7) Let $A_{1}, A_{2}, A_{3}, \cdots$ be subsets of a metric space. Show ...
Proof. Let $B_{n}=\cup_{i=1}^{n} A_{i}$. We prove part (a). First we show that $B_{n}^{\prime} \subset \cup_{i=1}^{n} A_{i}^{\prime}$. Suppose $p$ is a limit point of $B_{n}$. For the sake of contradiction suppose that $p$ is not a limit point for any $A_{i}$ or $p \notin \cup_{i=1}^{n} A_{i}^{\prime}$. Then for each $A_{i}$ there is some $r_{i}>0$ such that there does not exist a point $q$ where $q \neq p$ and $q \in N_{r_{i}}(p) \cap A_{i}$. Then take the smallest radius $r_{\min }$ out of all $r_{i}$ where $i \in\{1,2, \ldots, n\}$. Then for $r_{\text {min }}$ there does not exist a point $q \neq p$ such that $q \in N_{r_{\text {min }}}(p) \cap A_{i}$ for all $i \in\{1,2, \ldots, n\}$. Since $B_{n}=\cup_{i=1}^{n} A_{i}$ it follows $p$ is not a limit point of $B_{n}$. This is a contradiction. Hence, $p$ must be a limit point for some $A_{i}$ so $B_{n}^{\prime} \subset \cup_{i=1}^{n} A_{i}^{\prime}$.

Now we show $\cup_{i=1}^{n} A_{i}^{\prime} \subset B_{n}^{\prime}$. Suppose $p$ is a limit point for some $A_{i}$. By definition of a limit point, it follows for all $r>0$ there exists some $q \neq p$ such that $q \in N_{r}(p) \cap A_{i}$. Since $A_{i} \subset B_{n}$ it follows if $q \in N_{r}(p) \cap A_{i}$ then $q \in N_{r}(p) \cap B_{n}$. Thus, $p$ must also be a limit point of $B_{n}$. Hence, $\cup_{i=1}^{n} A_{i}^{\prime} \subset B_{n}^{\prime}$.

Since we have $\cup_{i=1}^{n} A_{i}^{\prime} \subset B_{n}^{\prime}$ and $B_{n}^{\prime} \subset \cup_{i=1}^{n} A_{i}^{\prime}$ it follows $B_{n}^{\prime}=\cup_{i=1}^{n} A_{i}^{\prime}$. That is $B_{n}$ and $\cup_{i=1}^{n} A_{i}$ share the same set of limit points. And since $B_{n}=\cup_{i=1}^{n} A_{i}$ it follows $\bar{B}_{n}=\cup_{i=1}^{n} \bar{A}_{i}$ Proof. Now we prove part (b).
Exercise. (8) Is every point of every open set $E \subset R^{2}$ a limit point of $E$ ? Answer the same question for closed sets in $R^{2}$.

Proof. Suppose $p \in E$ where $E$ is an open set. By definition of an open set it follows $p$ is an interior point. By definition of an interior point it follows for some $\delta>0$ we have $N_{\delta}(p) \subset E$. For the sake of contradiction suppose $p$ is not a limit point of $E$. Then there exists some $r>0$ such that there does not exist some point $q \neq p$ such that $q \in N_{r}(p) \cap E$. Now there are two cases to consider: either (1) $r \leq \delta$ or (2) $r>\delta$. Suppose we have case (1). If
$N_{\delta}(p) \subset E$ it follows for all $0<\epsilon \leq \delta$ that $N_{\epsilon}(p) \subset E$ since $N_{\epsilon}(p) \subset N_{\delta}(p)$. Since $0<r \leq \delta$ it follows $N_{r}(p) \subset E$. Then there exists some point $q \neq p$ such that $q \in N_{r}(p) \cap E$ which is a contradiction. Now suppose we have case (2). Then $r>\delta$. We have $N_{\delta}(p) \subset N_{r}(p)$. By definition of a neighborhood there exists some point $q \neq p$ such that $0<d(p, q)<\delta$. Since $N_{\delta}(p) \subset E$ it follows $q \in E$. Moreover, $q \in N_{r}(p)$. However, this contradicts the definition of $N_{r}(p)$. Since both cases lead to contradictions, it follows $p$ must be a limit point of $E$.

Exercise. (9) Let $E^{\circ}$ denote the set of all interior points of a set $E$. $E^{\circ}$ is called the interior of $E$.
(a) Prove that $E^{\circ}$ is always open.
(b) Prove that $E$ is open if and only if $E^{\circ}=E$.
(c) If $G \subset E$ and $G$ is open, prove that $G \subset E^{\circ}$.
(d) Prove that the complement of $E^{\circ}$ is the closure of the complement of $E$.
(e) Do $E$ and $\bar{E}$ always have the same interiors?
(f) Do $E$ and $E^{\circ}$ always have the same closures?

Proof. (a) We prove that $E^{\circ}$ is open. Let $p \in E^{\circ}$. It follows $p$ is an interior point of $E$. By definition of an interior point there exists some $r>0$ such that $N_{r}(p) \subset E$. Moreover, we know every neighborhood is open. Therefore, for all points $q \in N_{r}(p)$ we have $q$ is an interior point of $N_{r}(p)$. Since $N_{r}(p) \subset E$ it follows if $q$ is an interior point of $N_{r}(p)$ it is an interior point of $E$. Thus all points in $N_{r}(p)$ are interior points of $E$ so then $N_{r}(p) \subset E^{\circ}$. Then by definition of an interior point $p$ must be an interior point of $E^{\circ}$. The choice of $p$ was arbitrary: we only said $p \in E^{\circ}$. Hence, all points in $E^{\circ}$ are interior points. It follows $E^{\circ}$ is open.
Proof. (b) We prove that $E$ is open if and only if $E^{\circ}=E$. We first prove the forward direction. Suppose $E$ is open. Now we show $E \subset E^{\circ}$. Let $p \in E$. Since $E$ is open, it follows $p$ is an interior point. By definition of $E^{\circ}$ it follows $p \in E^{\circ}$. Thus $E \subset E^{\circ}$. Now we show $E^{\circ} \subset E$. Suppose $p \in E^{\circ}$. Then $p$ is an interior point. Then for some $r>0$ we have $N_{r}(p) \subset E$. Since $p \in N_{r}(p)$ it follows $p \in E$. Thus $E^{\circ} \subset E$. Since we have $E^{\circ} \subset E$ and $E \subset E^{\circ}$ we have $E=E^{\circ}$.

Now we prove the reverse direction. Suppose $E^{\circ}=E$. We show $E$ is open. Let $p \in E$. Since $E=E^{\circ}$ it follows $p$ is an interior points. This holds for all points $p$ in $E$. It follows $E$ is open.
Proof. (c) We prove that if $G \subset E$ and $G$ is open that $G \subset E^{\circ}$. Suppose $p \in G$. Since $G$ is open, there exists some $r>0$ such that $N_{r}(p) \subset G$. But we know $G \subset E$ so then $N_{r}(p) \subset E$. It follows $p$ is an interior point of $E$ and $E^{\circ}$ is the set of all interior points of $E$. It follows $p \in E^{\circ}$. Thus $G \subset E^{\circ}$.
Proof. (d) Prove that the complement of $E^{\circ}$ is the closure of the complement of $E$. First we show that $\left(E^{\circ}\right)^{c} \subset \overline{\left(E^{c}\right)}$. Suppose $p \in\left(E^{\circ}\right)^{c}$. Then $p \notin E^{\circ}$. It follows $p$ is not an interior point. Then for every $r>0$ we have there exists some $q \neq p$ such that $q \in N_{r}(p) \cap E^{c}$. For sake of contradiction suppose this were not true. Then we would have for some $r>0$ that for all $q \in N_{r}(p)$ that $q \in E$ so then $p$ would be an interior point. It follows our prior statement is true. Then by definition of a limit point, it follows $p$ is a limit point of $E^{c}$. Since $\overline{\left(E^{c}\right)}$ is the set of all limit points of $E^{c}$ it follows $p \in \overline{\left(E^{c}\right)}$. Thus $\left(E^{\circ}\right)^{c} \subset \overline{\left(E^{c}\right)}$.

Now we show that $\overline{\left(E^{c}\right)} \subset\left(E^{\circ}\right)^{c}$. Suppose $p \in \overline{\left(E^{c}\right)}$. Then $p$ is a limit point of $E^{c}$. Then for every $r>0$ there exists some $q \neq p$ such that $q \in N_{r}(p) \cap E^{c}$. Now for the sake of
contradiction suppose $p \notin\left(E^{\circ}\right)^{c}$. Then $p \in E^{\circ}$. Then there exists some $r>0$ such that for all $q \in N_{r}(p)$ we have $q \in E$ or in other words $N_{r}(p) \subset E$. But this contradicts the definition of $p$ being a limit point of $E^{c}$. Thus $p \in\left(E^{\circ}\right)^{c}$. It follows $\overline{\left(E^{c}\right)} \subset\left(E^{\circ}\right)^{c}$. We have shown both $\left(E^{\circ}\right)^{c} \subset \overline{\left(E^{c}\right)}$ and $\overline{\left(E^{c}\right)} \subset\left(E^{\circ}\right)^{c}$ so then $\left(E^{\circ}\right)^{c}=\overline{\left(E^{c}\right)}$.

Proof. (e) Do $E$ and $\bar{E}$ always have the same interiors? No. Suppose $Q \subset R$. Then $E^{\circ}=\varnothing$ and $(\bar{E})^{\circ}=R$.

Proof. (f) Do $E$ and $E^{\circ}$ have the same closures? No. Same example as part (e) disproves this. $\bar{E}=R$ and $\overline{\left(E^{\circ}\right)}=\varnothing$.

Exercise. (10) Let $X$ be an infinite set. For $p \in X$ and $q \in X$, define

$$
d(p, q)= \begin{cases}1 & (\text { if } p \neq q) \\ 0 & (\text { if } p=q)\end{cases}
$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

Proof. By definition of the metric we have $d(p, q)=0$ if $p=q$. If $p \neq q$ then we have $d(p, q)=1>0$. Also if $p=q$ then $d(p, q)=0=d(q, p)$. If $p \neq q$ then $d(p, q)=1=d(q, p)$. Now we show the metric satisfies the triangle inequality. That is, we show

$$
d(p, q) \leq d(p, r)+d(r, q)
$$

for all $p, r, q \in X$. There are two cases: either (1) $p=q$ or (2) $p \neq q$. Suppose we have case (1). Then $d(p, q)=0$. Then the inequality becomes

$$
0 \leq d(p, r)+d(r, q)
$$

which is true for all $p, r, q \in X$ since the $d(a, b) \geq 0$ for all $a, b \in X$. Now suppose we have case (2). Then $d(p, q)=1$. Then the inequality becomes

$$
1 \leq d(p, r)+d(r, q)
$$

Moreover, if $p \neq q$ then it is impossible to have both $r=p$ and $r=q$ for some $r \in X$ otherwise we would have $p=q$ which is a contradiction. Therefore, for any $r \in X$ we have that either $r \neq p$ or $r \neq q$ (or both). It follows in all three cases we have $d(p, r)+d(r, q) \geq 1$. Hence, the triangle inequality holds for all cases (1) and (2). Thus, this is a valid metric.

The empty set $\varnothing$ is open and closed vacuously. Now we find all open subsets of $X$. Suppose we have a nonempty set $E$. Let $p$ be a point in $E$. Then the neighborhood of radius 0.5 , $N_{0.5}(p)$ only contains the point $p$. Since $N_{0.5}(p)=p$ it follows $N_{0.5}(p) \in E$. Our choice of $p$ was arbitrary. Therefore, all points in $E$ for interior points and so $E$ is open. It follows all subsets of $X$ are open including $X$ itself.

Now we find all closed subsets of $X$. Suppose $E$ is a nonempty subset of $X$. To show $E$ is closed we must show all limit points of $E$ are contained in $E$. But we show that $E$ has no limit points. Suppose $p$ was a limit point of $E$. Then for every radius $r>0$ there exists some $q \neq p$ such that $q \in N_{r}(p) \cap E$. Let us choose $r=0.5$. Then $N_{0.5}(p)=\{p\}$. Thus, there are no points $q$ such that $q \neq p$ and $q \in N_{0.5}(p)$. It follows there are no limit points for any nonempty subset $E$ of $X$. Hence, all subsets are closed vacuously.

Now we show that only finite sets are compact. Suppose an infinite set was compact. We construct an open cover such that there does not exist any finite subcover. Let $E$ be an infinite subset of $X$. Then define an open cover $G$ as

$$
G=\cup_{p \in E} N_{0.5}(p)
$$

and we know each neighborhood $N_{0.5}(p)$ is open since we proved all subsets of $X$ are open. Moreover, $N_{0.5}(p)=\{p\}$. Hence, we need an infinite number of neighborhoods since each neighborhood corresponds to a single point $p \in E$. If we had chosen a finite number of neighborhoods, then we would have covered a corresponding finite number of points, yet there are an infinite quantity of points in $E$. Therefore, we must choose at minimum an infinite subcover. Thus, no infinite subset $E$ of $X$ is compact. However, finite subsets are compact. If we are given an infinite collection of open subsets $G$ that covers a finite set $K \subset X$, then we can choose a finite subcollection such that each open subset corresponds to some point in $K$.

Exercise. (11) For $x, y \in R$, define

$$
\begin{aligned}
d_{1}(x, y) & =(x-y)^{2} \\
d_{2}(x, y) & =\sqrt{|x-y|} \\
d_{3}(x, y) & =\left|x^{2}-y^{2}\right| \\
d_{4}(x, y) & =|x-2 y| \\
d_{5}(x, y) & =\frac{|x-y|}{1+|x-y|}
\end{aligned}
$$

(11) Determine, for each of these, whether it is a metric or not.

Proof. For each function we determine if it is a metric.
(1) The first function does not satisfy the triangle inequality.

$$
(x-y)^{2} \leq(x-z)^{2}+(z-y)^{2}
$$

Let $x=-1, z=0, y=1$. Then we have

$$
\begin{aligned}
(-1-1)^{2} & \leq(-1-0)^{2}+(0-1)^{2} \\
4 & \leq 1+1 \\
4 & \leq 2
\end{aligned}
$$

(2) The second function is symmetric. If $x=y$ then $d_{2}(x, y)=\sqrt{|x-y|}=0$. If $x \neq y$ then $|x-y|>0$ so $d_{2}(x, y)>0$. Now we show the function satisfies the triangle inequality. Generally, we know

$$
|p-q| \leq|p|+|q|
$$

for all $p, q \in R$. It follows

$$
|p-q| \leq|p|+\sqrt{|p||q|}+|q|
$$

Taking the square root of both sides gives

$$
\sqrt{|p-q|} \leq \sqrt{|p|}+\sqrt{|q|}
$$

Then let $p=x-z$ and $q=y-z$ for some $x, y, z \in R$. then we have

$$
\sqrt{|x-y|} \leq \sqrt{|x-z|}+\sqrt{|z-y|}
$$

but this is

$$
d_{2}(x, y) \leq d_{2}(x, z)+d_{2}(z, y)
$$

where $x, y, z \in R$. It follows $d_{2}$ is a metric.
(3) The third function is not a metric. Let $x=-1$ and $y=1$. Then $d_{3}(-1,1)=|1-1|=$ 0 even though $x \neq y$.
(4) The fourth function is not symmetric. For some $x, y \in R$ we have $d_{4}(x, y) \neq d_{4}(y, x)$.
(5) We show the fifth function is a metric. First if $x=y$ then $d(x, y)=0 /(1+0)=0$. If $x \neq y$ then we have $|x-y|>0$ so then $d_{5}(x, y)>0 . d_{5}(x, y)$ is symmetric since $|x-y|$ is symmetric. And now we show $d_{5}$ satisfies the triangle inequality. We know in general for all $p, q \in R$

$$
|p-q| \leq|p|+|q|
$$

Then, we add $2|p||q|+|p||q||p-q|$ on the right hand side.

$$
|p-q| \leq|p|+|q|+2|p||q|+|p||q||p-q|
$$

Then we add $|p-q \| p|$ to both sides.

$$
|p-q|+|p-q||p| \leq|p|+|p-q||p|+|q|+2|p||q|+|p||q||p-q|
$$

Then we factor.

$$
|p-q|(1+|p|) \leq|p|(1+|p-q|)+|q|+2|p||q|+|p||q||p-q|
$$

Then we add $|p-q \| q|$ to both sides.
$|p-q|(1+|p|)+|p-q||q| \leq|p|(1+|p-q|)+|q|+|p-q||q|+2|p||q|+|p||q||p-q|$
Then we factor.

$$
|p-q|(1+|p|+|q|) \leq|p|(1+|p-q|)+|q|(1+|p-q|)+2|p||q|+|p||q||p-q|
$$

Then we factor again.

$$
|p-q|(1+|p|+|q|) \leq|p|(1+|p-q|+|q|)+|q|(1+|p-q|+|p|)+|p||q||p-q|
$$

Then we add $|p\|q\| p-q|$ to both sides.

$$
|p-q|(1+|p|+|q|)+|p||q||p-q| \leq|p|(1+|p-q|+|q|)+|q|(1+|p-q|+|p|)+2|p||q||p-q|
$$

Then we factor.
$|p-q|(1+|p|+|q|+|p||q|) \leq|p|(1+|p-q|+|q|+|q||p-q|)+|q|(1+|p-q|+|p|+|p||p-q|)$
And factor again.

$$
|p-q|(1+|p|)(1+|q|) \leq|p|(1+|q|)(1+|p-q|)+|q|(1+|p|)(1+|p-q|)
$$

Then we divide by $(1+|p|)(1+|q|)(1+|p-q|)$ on both sides.

$$
\frac{|p-q|}{1+|p-q|} \leq \frac{|p|}{1+|p|}+\frac{|q|}{1+|q|}
$$

Then let $p=x-z$ and $q=y-z$ for some $x, y, z \in R$

$$
\frac{|x-y|}{1+|x-y|} \leq \frac{|x-z|}{1+|x-z|}+\frac{|y-z|}{1+|y-z|}
$$

Then rewrite in terms of $d_{5}$.

$$
d_{5}(x, y) \leq d_{5}(x, z)+d_{5}(z, y)
$$

Thus $d_{5}$ is a metric.

Exercise. (12) Let $K \subset R^{1}$ consist of 0 and numbers $1 / n$, for $n=1,2,3, \cdots$. Prove that $K$ is compact directly from the definition (without using Heine-Borel theorem).

Proof. Suppose we some subset $K$ of $R^{1}$ have an open cover $\left\{G_{\alpha}\right\}$ such that $K \subset \cup_{\alpha} G_{\alpha}$. In order to show $K$ is compact we must show that there is some finite subcover. We prove by contradiction. Suppose there is not a finite subcover. Let $E_{m}=\{1 / n \mid n=1,2,3, \ldots, m\}$. Since $E_{m}$ is finite, there is a finite subcover for $E_{m}$. It follows $K-E_{m}$ cannot be covered by any finite subcollection of $\left\{G_{\alpha}\right\}$. We know $0 \in K$ and thus $0 \in G_{\alpha}$ for some $\alpha$. Moreover, since $G_{\alpha}$ is an open subset of $R^{1}$ there exists $r>0$ such that $N_{r}(0) \subset G_{\alpha}$. Equivalently, for some $r>0$ for all $y \in R^{1}$, if $|0-y|=|y|<r$ then $y \in G_{\alpha}$. Let $w$ be defined as the largest natural number $m$ such that $r \geq 1 / m$. It follows $r \leq x$ for all $x \in E_{w}$. Then $\frac{1}{w+1}<r$ otherwise $\frac{1}{w+1} \geq r$ which contradicts the definition of $w$. Let $T=K-E_{w}$. Then the set $T$ is defined as

$$
T=\{0\} \bigcup\left\{\frac{1}{w+1}, \frac{1}{w+2}, \cdots\right\}
$$

It follows for all $x \in T$ we have $0 \leq x<r$. Thus $|x|<r$ for all $x \in T$. Then $T=K-E_{w} \subset$ $G_{\alpha}$. But $K-E_{w}$ cannot be covered by some finite subcover. This is a contradiction. Therefore, the set $K$ is compact.

Exercise. (13) Construct a compact set of real numbers whose limit points form a countable set.

Exercise. (14) Give an example of an open cover of the segment $(0,1)$ which has no finite subcover.

Exercise. (15) Show that Theorem 2.36 and its Corollary become false if the word "compact" is replaced by "closed" or by "bounded."

Exercise. (16) Regard $Q$, the set of rational numbers, as a metric space, with $d(p, q)=|p-q|$. Let $E$ be the set of all $p \in Q$ such that $2<p^{2}<3$. Show that $E$ is closed and bounded in $Q$, but that $E$ is not compact. Is $E$ open in $Q$ ?

Proof. We first show that $E$ is closed. It is sufficient to show if a point $x$ is a limit point of $E$ then $x$ is in $E$. However, we prove the contrapositive, that is, we show if $x$ is not in $E$
then $x$ is not a limit point of $E$. Note that $E$ is defined as

$$
\begin{aligned}
E & =\left\{p \in \mathbb{Q} \mid 2<p^{2}<3\right\} \\
& =\{p \in \mathbb{Q} \mid-\sqrt{3}<p<-\sqrt{2} \text { or } \sqrt{2}<p<\sqrt{3}\}
\end{aligned}
$$

Suppose $q \notin E$. Then we have $q<-\sqrt{3},-\sqrt{2}<q<\sqrt{2}$, or $\sqrt{3}<q$. Without a loss of generality, consider the case where $-\sqrt{2}<q<\sqrt{2}$. Then there exist $\varepsilon \in \mathbb{R}$ such that $\varepsilon>0$ and

$$
-\sqrt{2}<q-\varepsilon<q<q+\varepsilon<\sqrt{2}
$$

Then we have $N_{\varepsilon}(q)=\left\{w \in \mathbb{Q}|d(q, w)=|q-w|<\varepsilon\}\right.$. It follows that if $a \in N_{\varepsilon}(q)$ then $-\sqrt{2}<a<\sqrt{2}$. Hence, $a \notin E$ so then $N_{\varepsilon}(q) \subset E^{c}$. It follows $N_{\varepsilon}(q) \cap E=\varnothing$. Hence, $q$ cannot be a limit point of $E$. The same argument applies for when $q<-\sqrt{3}$ and $q>\sqrt{3}$. Hence, we proved the contrapositive of the desired statement. Therefore, $E$ is closed.

Now we show $E$ is bounded. Since $E$ is defined as

$$
E=\{p \in \mathbb{Q} \mid-\sqrt{3}<p<-\sqrt{2} \text { or } \sqrt{2}<p<\sqrt{3}\}
$$

it follows for all $x \in E$

$$
-2=-\sqrt{4}<-\sqrt{3}<x<\sqrt{3}<\sqrt{4}=2
$$

Hence $d(0, x)=|x| \leq 2$. Hence, $E$ is bounded.
Now we show that $E$ is not compact. We construct an open cover of $E$ such that there does not exist a finite subcover. By Theorem 1.20 part (b) we know there exist rational numbers $a_{n}, b_{n} \in \mathbb{Q}$ such that

$$
\begin{aligned}
-\sqrt{3} & <-b_{n}<-\sqrt{3}+1 / n \\
-\sqrt{2}-1 / n & <-a_{n}<-\sqrt{2} \\
\sqrt{2} & <a_{n}<\sqrt{2}+1 / n \\
\sqrt{3}-1 / n & <b_{n}<\sqrt{3}
\end{aligned}
$$

for all $n \in \mathbb{N}$. Moreover, for all $n \geq 7$ we have

$$
-b_{n}<-a_{n}<a_{n}<b_{n}
$$

Note that if $n<7$ then the strict inequality $a_{n}<b_{n}$ does not hold. Then for $n \geq 7$ take $I_{n}=\left(-b_{n},-a_{n}\right) \cup\left(a_{n}, b_{n}\right) \subset \mathbb{Q}$ where

$$
\begin{aligned}
\left(a_{n}, b_{n}\right) & =\left\{x \in \mathbb{Q} \mid a_{n}<x<b_{n}\right\} \\
\left(-b_{n}, a_{n}\right) & =\left\{x \in \mathbb{Q} \mid-b_{n}<x<-a_{n}\right\}
\end{aligned}
$$

Then let $I=\cup_{n=7}^{\infty} I_{n}$. Note that $(a, b)=\{x \in \mathbb{Q} \mid a<x<b\}$ where $a, b \in \mathbb{Q}$ is an open set since $N_{r}(p)=(a, b)$ where $r=(b-a) / 2$ and $p=(a+b) / 2 \in \mathbb{Q}$. Hence each set $\left(a_{n}, b_{n}\right)$ is a neighborhood which is open by Theorem 2.19. And $I_{n}=\left(-b_{n},-a_{n}\right) \cup\left(a_{n}, b_{n}\right)$ is a union of open sets which is open by Theorem 2.28. Then $I$ is a arbitrary union of open sets which is open by Theorem 2.28. Now we show that $I$ is an open cover of $E$.

We show that $E \subset I$. Let $x \in E$. Then $\sqrt{2}<|x|<\sqrt{3}$. Then by the archimedean property there exists sufficiently large natural numbers $m_{1}$ and $m_{2}$ such that

$$
\sqrt{2}<\sqrt{2}+1 / m_{1}<|x|<\sqrt{3}-1 / m_{2}<\sqrt{3}
$$

Then take $m=\max \left\{m_{1}, m_{2}\right\}$. Then we have

$$
\sqrt{2}+1 / m<|x|<\sqrt{3}-1 / m
$$

It follows $x \in I_{m}$ hence $x \in I$. It follows $E \subset I$.
Now we show there does not exist a finite subcover for the collection of open sets $\left\{I_{\alpha}\right\}_{\alpha \in \mathbb{N}, \alpha \geq 7}$. For the sake of contradiction, suppose there were a finite subcover. Then we have

$$
E \subset I_{\alpha_{1}} \cup I_{\alpha_{2}} \cup \cdots \cup I_{\alpha_{n}}
$$

Where $\left\{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}\right\} \subset\{7,8,9,10,11, \ldots\}$. Since $\left\{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}\right\}$ is finite, take the maximum of the set. Take $M=\max \left\{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}\right\}$. Since $\alpha_{i} \leq M$ for all $\alpha_{i} \in\left\{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}\right\}$ it follows

$$
\sqrt{2}<\sqrt{2}+\frac{1}{M} \leq \sqrt{2}+\frac{1}{\alpha_{i}} \text { and } \sqrt{3}-\frac{1}{\alpha_{i}} \leq \sqrt{3}-\frac{1}{M}<\sqrt{3}
$$

for all $\alpha_{i}$. Hence, $I_{m} \supset I_{\alpha_{i}}$ for all $\alpha_{i}$. It follows

$$
I_{M}=I_{\alpha_{1}} \cup I_{\alpha_{2}} \cup \cdots \cup I_{\alpha_{n}}
$$

and then

$$
E \subset I_{M}
$$

By Theorem 1.20 there is a rational $q \in \mathbb{Q}$ such that $\sqrt{3}-1 / M<q<\sqrt{3}$. Hence $q \notin I_{M}$ but $q \in E$. This is a contradiction. Therefore, there does not exist a finite subcover for the open cover $\left\{I_{n}\right\}_{n \in \mathbb{N}, n \geq 7}$. Hence, $E$ is not compact.

Now we show that $E$ is open. Let $x \in E$. Then $x \in \mathbb{Q}$ and $2<x^{2}<3$. Then $\sqrt{2}<|x|<\sqrt{3}$. Then take $r=\min \{\sqrt{3}-|x|,|x|-\sqrt{2}\}$. Then $N_{r}(x) \subset E$. Hence, $E$ is open.

Exercise (17). Let $E$ be the set of all $x \in[0,1]$ whose decimal expansion contains only the digits 4 and 7 . Is $E$ countable? Is $E$ dense in $[0,1]$ ? Is $E$ compact? Is $E$ perfect?
Proof. $E$ is not countable. There is a bijection between the set of sequences of all 4's and 7's with the set of all sequences of 0's and 1's which we showed was uncountable. It has a cardinality of $\left|2^{\mathbb{N}}\right|=|\mathbb{R}|$ which is uncountable. $E$ is not dense, since the largest number in $E$ is $0.777 \ldots$ so then 1 is not a limit point of $E$ since we can choose a neighborhood such as $N_{0.1}(1)$ which does not intersect $E$ since $1-0.1>0.777 \ldots$.
$E$ is not compact. Take the open cover $\left\{G_{n}\right\}_{n \in \mathbb{N}}$ where $G_{n}=\left(0,0.7_{n}\right)$ where $0.7_{n}$ denotes the digit 7 repeated $n$ times. Equivalently let $s_{n}=\sum_{i=1}^{n} 7 \cdot 10^{-i}$. Then $G_{n}=\left(0, s_{n}\right)$. For the sake of contradiction suppose there were a finite subcover
Exercise (18). Consider the cantor set. It is a perfect subset of $\mathbb{R}$. Every element in the cantor set is rational. Add a value say $\pi$ to every number. Then we have a perfect subset of $\mathbb{R}$ that is a subset of the irrationals.

Exercise (19). There are five parts.
(a) If A and B are disjoint closed sets in some metric space $X$, prove that they are separated.
(b) Prove the same for disjoint open sets.
(c) Fix $p \in X, \delta>0$, define $A$ to the set the of all $q \in X$ for which $d(p, q)<\delta$, define $B$ similarly with $>$ in place of $<$. Prove that $A$ and $B$ are separated.
(d) Prove that every connected metric space with at least two points is uncountable.

Proof. We prove part (a) first. Note by definition of closure that $A \subset \bar{A}$ and $B \subset \bar{B}$. Then since $\bar{A} \cap \bar{B}=\varnothing$ it follows $A \cap \bar{B}=\varnothing$ and $\bar{A} \cap B=\varnothing$.

We prove part (b). Suppose we have two open disjoint sets $A$ and $B$. We show they are separated. We show that $\bar{A} \cap B=\varnothing$. For the sake of contradiction, suppose $p \in \bar{A} \cap B$. Then $p \in B$. Since $B$ is open it follows $N_{r}(p) \subset B$. Note $p \in \bar{A}$ but $p \notin A$ since $A \cap B=\varnothing$. It follows $p$ is a limit point of $A$. Then by definition of a limit point, we have $\left(N_{r}(p)-\{p\}\right) \cap A=$ $\varnothing$. Let $q \in\left(N_{r}(p)-\{p\}\right) \cap A$. Then $q \in A$. But since $q \in N_{r}(p)$ and $N_{r}(p) \subset B$ it follows $q \in B$. Then $q \in A \cap B$ which is a contradiction. It follows $\bar{A} \cap B=\varnothing$. The same argument shows that $A \cap \bar{B}=\varnothing$. It follows $A$ and $B$ are separated sets.

We prove part (c). Let $A=\{x \in X \mid d(p, x)<\delta\}$ and $B=\{x \in X \mid d(p, x)>\delta\}$. We show $A$ and $B$ are separated. By part $B$ it is sufficient to show $A$ and $B$ are disjoint open sets. For the sake of contradiction, suppose $A$ and $B$ are not disjoint. Let $q \in A \cap B$. Then $d(p, q)<\delta$ and $d(p, q)>\delta$ which is a contradiction. It follows $A \cap B=\varnothing$. We know $A$ is a neighborhood $N_{\delta} p$ so it is open which since we previously prove that. Now we show $B$ is open. Let $q \in B$. Fix $q$. Then $d(p, q)>\delta$. Then let $\varepsilon=d(p, q)-\delta$. Then take $w \in N_{\varepsilon}(q)$. Then $d(w, q)<\varepsilon$. Then

$$
\begin{aligned}
-\varepsilon & <-d(w, q) \\
-\varepsilon+d(p, q) & <d(p, q)-d(w, q) \\
-(d(p, q)-\delta)+d(p, q) & <d(p, q)-d(w, q) \\
\delta & <d(p, q)-d(w, q)
\end{aligned}
$$

Moreover, by the triangle inequality we have

$$
\begin{aligned}
d(p, q) & \leq d(p, w)+d(w, q) \\
d(p, q)-d(w, q) & \leq d(p, w)
\end{aligned}
$$

Putting these two inequalities together we get

$$
\delta<d(p, q)-d(w, q) \leq d(p, w)
$$

It follows $w \in B$. It follows $N_{\varepsilon}(q) \subset B$. Then $B$ is open. Since $A$ and $B$ are both open, by part B it follows $A$ and $B$ are separated.

Now we prove part (d). Consider the set $C$ which contains at least two points $p$ and $q$ and is connected. For some $0<\delta<d(p, q)$ define the sets

$$
\begin{aligned}
& A=\{x \in X \mid d(p, x)<\delta\} \\
& B=\{x \in X \mid d(p, x)>\delta\}
\end{aligned}
$$

By part (c) we know that these sets are separated. Since $C$ is connected, there must exist some point $w \in C-(A \cup B)$ such that $d(p, w)=\delta$. This holds for all $\delta \in \mathbb{R}$ such that $0<\delta<d(p, q)$. Consider the set

$$
D_{r}=\{w \in C \mid d(p, w)<r\}
$$

where $0<r<d(p, q)$. Note that if $r_{1} \neq r_{2}$ then $D_{r_{1}} \cap D_{r_{2}}=\varnothing$. By the axiom of choice there is a choice function

$$
f:(0, d(p, q)) \rightarrow \cup_{r \in(0, d(p, q))} D_{r}
$$

such that $f(r) \in D_{r}$ for all $r$ and where $(0, d(p, q)) \subset \mathbb{R}$ is an open interval in $\mathbb{R}$. It follows, we have an injection from the interval $(0, d(p, q))$ to $\cup D_{r} \subset C$. Hence, $|(0, d(p, q))| \leq|C|$. But we know the open interval $(0, d(p, q))$ is uncountable so then $C$ must be uncountable.

Exercise (19). Let $A$ and $B$ be separated subsets of $\mathbb{R}^{k}$, suppose $\boldsymbol{a} \in A, \boldsymbol{b} \in B$, and define

$$
\boldsymbol{p}(t)=(1-t) \boldsymbol{a}+t \boldsymbol{b}
$$

for $t \in \mathbb{R}$. Put $A_{0}=\boldsymbol{p}^{-1}(A), B_{0}=\boldsymbol{p}^{-1}(B)$.
(a) Prove that $A_{0}$ and $B_{0}$ are separated subsets of $\mathbb{R}$.
(b) Prove that there exists $t_{0} \in(0,1)$ such that $\boldsymbol{p}\left(t_{0}\right) \notin A \cup B$.
(c) Prove that every convex subset of $\mathbb{R}^{k}$ is connected.

Proof. We prove part (a) first. We prove by contradiction. Without a loss of generality let $t_{0} \in \bar{A}_{0} \cap B_{0}$. We show that $\boldsymbol{p}\left(t_{0}\right) \in \bar{A} \cap B$. There are two cases to consider. If $t_{0} \in A_{0} \cap B_{0}$ then $\boldsymbol{p}\left(t_{0}\right) \in A \cap B$.

Now consider the second case where $t_{0} \in B_{0}-A_{0}$ and $t_{0}$ is a limit point of $A_{0}$. We show that $\boldsymbol{p}\left(t_{0}\right)$ is a limit point of $A$. It is sufficient to show that for any $\varepsilon>0$ we have

$$
\left(N_{\varepsilon}\left(\boldsymbol{p}\left(t_{0}\right)\right)-\left\{\boldsymbol{p}\left(t_{0}\right)\right\}\right) \cap A \neq \varnothing
$$

Then take $\delta=\varepsilon /|\boldsymbol{a}-\boldsymbol{b}|$. Since $t_{0} \notin A_{0}$ and is a limit point of $A_{0}$ there exists a $t_{1} \in A_{0}$ such that $\left|t_{0}-t_{1}\right|<\delta$. Then we have

$$
\begin{aligned}
\left|\boldsymbol{p}\left(t_{0}\right)-\boldsymbol{p}\left(t_{1}\right)\right| & =\left|\left(1-t_{0}\right) \boldsymbol{a}+t_{0} \boldsymbol{b}-\left(1-t_{1}\right) \boldsymbol{a}-t_{1} \boldsymbol{b}\right| \\
& =|\boldsymbol{a}-\boldsymbol{b}|\left|t_{0}-t_{1}\right|
\end{aligned}
$$

since $\left|t_{0}-t_{1}\right|<\delta$ and by definition of $\delta$ we have

$$
\left|\boldsymbol{p}\left(t_{0}\right)-\boldsymbol{p}\left(t_{1}\right)\right|<\varepsilon
$$

Our choice of $\varepsilon$ was arbitrary. It follows $\left(N_{\varepsilon}\left(\boldsymbol{p}\left(t_{0}\right)\right)-\left\{\boldsymbol{p}\left(t_{0}\right)\right\}\right) \cap A \neq \varnothing$ for all $\varepsilon>0$. Hence, $\boldsymbol{p}\left(t_{0}\right)$ is a limit point of $A$. Since $t_{0} \in B_{0}$ we have $\boldsymbol{p}\left(t_{0}\right) \in B$. Then $\boldsymbol{p}\left(t_{0}\right) \in \bar{A} \cap B$. This contradicts $A$ and $B$ being separated sets. Hence, $A_{0}$ and $B_{0}$ must be separated.

Now we prove part (b). We prove by contradiction. Suppose for all $t_{0} \in[0,1]$ that $\boldsymbol{p}\left(t_{0}\right) \in A \cup B$. We know that $\boldsymbol{p}(0)=\boldsymbol{a}$ and $\boldsymbol{p}(1)=\boldsymbol{b}$ so then $A_{0}$ is bounded above. Specifically for all $a \in A_{0}$ we have $a<1$. Since $A_{0}$ is bounded above and is a subset of $\mathbb{R}$ it follows there is a least upper bound. Take $\alpha=\sup A_{0}$. There are two cases, either $\alpha \in A_{0}$ or $\alpha \in B_{0}$ since for all $t_{0} \in[0,1]$ we have $\boldsymbol{p}\left(t_{0}\right) \in A \cup B$.

Consider the case where $\alpha \in A_{0}$. Take $\varepsilon>0$. Then consider the interval $(\alpha-\varepsilon, \alpha+\varepsilon)$. Since $\alpha$ is the supremum of $A_{0}$ and for all $t_{0} \in[0,1]$ we have $\boldsymbol{p}\left(t_{0}\right) \in A \cup B$, it follows for all $1 \geq t_{0}>\alpha$ that $\boldsymbol{p}\left(t_{0}\right) \in B$ and $t_{0} \in B_{0}$. There exists $\beta \in \mathbb{R}$ such that $\alpha<\beta<\alpha+\varepsilon$ and since $\beta>\alpha$ we have $\beta \in B_{0}$. Hence, $\left(N_{\varepsilon}(\alpha)-\{\alpha\}\right) \cap B_{0} \neq \varnothing$. Thus, $\alpha$ is a limit point of $B_{0}$. Then $\alpha \in A_{0} \cap \bar{B}_{0}$.

Now consider the case where $\alpha \notin A_{0}$. Then $\alpha \in B_{0}$. Now we show $\alpha$ is a limit point of $A_{0}$. Since $\alpha=\sup A_{0}$, for any $\varepsilon>0$ consider the interval $(\alpha-\varepsilon, \alpha+\varepsilon)$. Since $\alpha \notin A_{0}$ and by definition of supremum, there exists $\alpha^{\prime}$ such that $\alpha-\varepsilon<\alpha^{\prime}<\alpha$ and $\alpha^{\prime} \in A_{0}$. This is true for all $\varepsilon>0$. Hence, $\left(N_{\varepsilon}(\alpha)-\{\alpha\}\right) \cap A_{0} \neq \varnothing$. Thus, $\alpha$ is a limit point of $A_{0}$. So then $\alpha \in \bar{A}_{0} \cap B_{0}$.

In part (a) we showed that $A_{0}$ and $B_{0}$ are separated sets. Yet we just showed that either $A_{0} \cap \bar{B}_{0} \neq \varnothing$ or $\bar{A}_{0} \cap B_{0} \neq \varnothing$ which implies $A_{0}$ and $B_{0}$ are not separated. This is a contradiction. Hence, there exists $t_{0} \in(0,1)$ such that $p\left(t_{0}\right) \notin A \cup B$.

Now we prove part (c). We show that every convex subset of $\mathbb{R}^{k}$ is connected. For the sake of contradiction, suppose there exists a convex set $C$ in $\mathbb{R}^{k}$ that is not connected. Then
$C$ is a union of two separated sets, say $A$ and $B$. Then take $\boldsymbol{a} \in A$ and $\boldsymbol{b} \in B$. Then consider the function $\boldsymbol{p}:(0,1) \rightarrow \mathbb{R}^{k}$ defined as

$$
\boldsymbol{p}(t)=(1-t) \boldsymbol{a}+t \boldsymbol{b}
$$

By part (b), there exists some $t_{0} \in(0,1)$ such that $\boldsymbol{p}\left(t_{0}\right) \notin A \cup B$. Since $C=A \cup B$ then we have for some $t_{0}$ that $\boldsymbol{p}\left(t_{0}\right) \notin C$. But since $C$ is convex, we must have for all $t_{0} \in(0,1)$ that $\boldsymbol{p}\left(t_{0}\right) \in C$. This is a contradiction. Therefore, all convex sets are connected.
Exercise (22). A metric space is called separable if it contains a countable dense subset. Show that $\mathbb{R}^{k}$ is separable.

Proof. Consider the set $\mathbb{Q}^{k}$. It is a subset of $\mathbb{R}^{k}$ and is a finite cartesian product of countable sets which is countable. Now we have to show $\mathbb{Q}^{k}$ is dense in $\mathbb{R}^{k}$. Take $\varepsilon>0$ and $x \in \mathbb{R}^{k}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ for every $x_{i} \in \mathbb{R}$ there exists a $p_{i} \in \mathbb{Q}$ such that $x_{i} \neq p_{i}$ and

Then we have

$$
\left|p_{i}-x_{i}\right|<\frac{\varepsilon}{\sqrt{k}}
$$

then we have

$$
\begin{aligned}
&\left|p_{i}-x_{i}\right|^{2}<\frac{\varepsilon^{2}}{k} \\
& \sum_{i=1}^{k}\left(p_{i}-x_{i}\right)^{2}<\varepsilon^{2} \\
&\left(\sum_{i=1}^{k}\left(p_{i}-x_{i}\right)^{2}\right)^{(1 / 2)}<\varepsilon
\end{aligned}
$$

Let $p=\left(p_{1}, p_{2}, \ldots, p_{k}\right)$. It follows $p \in N_{\varepsilon}(x)-\{x\}$. Hence, $x$ is a limit point of $\mathbb{Q}$ for all $x \in \mathbb{R}$. Hence $\mathbb{R}^{k}$ is separable.

Exercise (23). Prove that every separable metric space has a countable base.
Proof. Let $(X, d)$ be a separable metric space. Since our metric space is separable there exists a countable dense subset of $X$, say $E$. Then define $\mathcal{B}$ as the collection of all neighborhoods of rational radii for all points in $E$. It follows there is a bijection $f: E \times \mathbb{Q} \rightarrow \mathcal{B}$. Since $E$ is countable and $\mathbb{Q}$ is countable, it follows $E \times \mathbb{Q}$ is countable. Hence, $\mathcal{B}$ is countable.

Now we show that $\mathcal{B}$ is a base for our metric space $(X, d)$. Let $U$ be an open set and let $p$ be a point in $U$. By definition of an open set, it follows there exists some $\varepsilon>0$ such that $N_{\varepsilon}(p) \subset U$. There are two cases, either $p \in E$ or $p \notin E$.

If $p \in E$ then take $\delta \in \mathbb{Q}$ as $0<\delta<\varepsilon$. Then $N_{\delta}(p)$ is a open set in $\mathcal{B}$. And $p \in N_{\delta}(p) \subset$ $N_{\varepsilon}(p) \subset U$.

Now consider the case where $p \notin E$. Then let $\delta \in \mathbb{Q}$ be constrained to $0<\delta<\varepsilon$. Since $E$ is a dense subset of $X$, it follows $p$ is a limit point of $E$. Then there exists a point $q \in E$ such that $d(p, q)<\delta / 2$. Now we show that $N_{\delta / 2}(q) \subset N_{\varepsilon}(p)$. Let $w \in N_{\delta / 2}(q)$. Then $d(q, w)<\delta / 2$. Moreover, we know by our choice of $q$ we have $d(p, q)<\delta / 2$. Then by the triangle inequality we have

$$
d(p, w) \leq d(p, q)+d(q, w)<\delta / 2+\delta / 2=\delta<\varepsilon
$$

Hence, $d(p, w)<\varepsilon$ and $w \in N_{\varepsilon}(p)$. It follows $N_{\delta / 2}(q) \subset N_{\varepsilon}(p)$ so then $p \in N_{\delta / 2}(q) \subset U$. Thus, $\mathcal{B}$ is a countable base for our separable metric space $(X, d)$.

Exercise (24). Let $X$ be a metric space in which every infinite subset has a limit point. Prove that $X$ is separable.

Proof. Let $X$ be a metric space in which every infinite subset has a limit point. Suppose we try to construct a set by the following method. Fix $\delta>0$. Then pick $x_{1} \in X$. Then having chosen $x_{1}, \ldots, x_{j} \in X$, choose $x_{j+1} \in X$, if possible, so that $d\left(x_{i}, x_{j+1}\right) \geq \delta$ for $i=1, \ldots, j$. Suppose we could iterate this process indefinitely. Then we would have constructed an infinite subset of $X$ that has no limit points, since all points are isolated. This contradicts the assumption that every infinite subset has a limit point. Hence, our construction of a set of points in $X$ iterates only a finite number of times. Hence, $X$ can be covered by finitely many neighborhoods of radius $\delta$. We use this fact to show $X$ must be separable.

Define the set $S_{k}=\left\{x_{1}^{k}, x_{2}^{k}, \ldots, x_{n_{k}}^{k}\right\}$ such that

$$
X \subset N_{1 / k}\left(x_{1}^{k}\right) \cup N_{1 / k}\left(x_{2}^{k}\right) \cup \cdots \cup N_{1 / k}\left(x_{n_{k}}^{k}\right)
$$

where $k \in \mathbb{N}$. Then $S_{k}$ has a total number of $n_{k}$ distinct points such that neighborhoods of radius $1 / k$ cover $X$. Then define $S=\cup_{i=1}^{\infty} S_{i}$. Now we show $S$ is a countable dense subset of $X$. Let $x \in X$. If $x \in S$ then we are done. Now consider the case where $x \notin S$. We show $x$ must be a limit point of $S$. Take $\varepsilon>0$ and $N_{\varepsilon}(x)$. By the archimedean property there exists a sufficiently large $n \in \mathbb{N}$ such that $0<1 / n<\varepsilon$. Fix $n$. Then there exists a point $p \in S_{n}$ such that $x \in N_{1 / n}(p)$. Since we assumed $x \notin S$ it follows $x \neq p$. Then we have $d(x, p)<1 / n<\varepsilon$. Then $\left(N_{\varepsilon}(x)-\{x\}\right) \cap S \neq \varnothing$. This holds for all $\varepsilon$ since our choice of $\varepsilon$ was arbitrary. It follows that $x$ is a limit point of $S$. Therefore, $S$ is a dense subset of $X$. Moreover, $S$ consists of a countable union of finite sets which is countable. Hence, $S$ is a countable dense subset of $X$. It follows $X$ is a separable metric space.

Exercise (25). Prove that every compact metric space $K$ has a countable base, and that $K$ is therefore separable.

Proof. Take an open cover of $K$ as $\cup_{x \in K} N_{1 / n}(x)$. Since $K$ is compact there is a finite subcover

$$
K \subset N_{1 / n}\left(x_{1}\right) \cup N_{1 / n}\left(x_{2}\right) \cup \cdots \cup N_{1 / n}\left(x_{m}\right)
$$

Let $S_{n}$ be the set of points that correspond to the centers of the neighborhoods of radius $1 / n$ of a finite subcover of $K$ such that $\left\{x_{1}, x_{2}, \ldots, x_{m}\right\}$ in the previous example. Then take $S=\cup_{i=1}^{\infty} S_{i}$.

Now we show $S$ is a countable dense subset of $K . S$ is countable since it is a countable union of a finite set of points. Now we show $S$ is dense in $K$. Let $p \in K$. If $p \in S$ we are done. Now consider $p \notin S$ and some arbitrary neighborhood $N_{\varepsilon}(p)$ where $\varepsilon>0$. Then there exists a sufficiently large $n$ such that $0<1 / n<\varepsilon$. It follows there exists some $x \in S_{n}$ such that $d(x, p)<1 / n$. Hence $x \in\left(N_{\varepsilon}(p)-\{p\}\right) \cap S$. Our choice of $\varepsilon$ was arbitrary. It follows $\left(N_{\varepsilon}(p)-\{p\}\right) \cap S \neq \varnothing$ for all $\varepsilon$. Hence, $p$ is a limit point of $S$. Then $S$ is a countable dense subset of $K$. It follows $K$ is a separable metric space. Moreover, by exercise 23 it follows $K$ has a countable base.

Exercise (26). Let $X$ be a metric space in which every infinite subset has a limit point. Prove that $X$ is compact.

Proof. Take $\left\{G_{\alpha}\right\}$ to be an open cover of $X$. Since every infinite subset has a limit point, it follows $X$ has a countable subcover. Then $X \subset G_{1} \cup G_{2} \cup \cdots$. For the sake of contradiction, suppose there is no finite subcover. Then take $F_{n}=\left(G_{1} \cup G_{2} \cup \cdots \cup G_{n}\right)^{c}$. It follows $F_{n}$ is non-empty for all $n \in \mathbb{N}$ since there is no finite subcover. For each $F_{n}$ take $x_{n} \in F_{n}$. Then let $E=\cup_{i=1}^{\infty} x_{i}$. Then $E$ is an infinite subset and by the hypothesis there exists some limit point of $E$, say $q$. Since $X \subset G_{1} \cup G_{2} \cup \cdots$ it follows $q \in G_{j}$ for some $j$. Note that $G_{j}$ is open so there exists some $\varepsilon>0$ such that $N_{\varepsilon}(q) \subset G_{j}$. Moreover, since $q$ is a limit point, it follows the neighborhood $N_{\varepsilon}(q)$ must contain an infinite number of points of $E$. Therefore, an infinite number of points are in $G_{j}$. Note that by nature of construction of $E$, all points $x_{k} \notin G_{j}$ for all $k \geq j$ since $x_{k} \in\left(G_{1} \cup \cdots \cup G_{j} \cup \cdots G_{k}\right)^{c}$. It follows all but a finite number of points are in $G_{j}$. Specifically, at most the points $x_{1}, x_{2}, \ldots, x_{j-1}$ are in $G_{j}$. This is a contradiction. Hence, the metric space $X$ is compact.

Exercise (27). Define a point $p$ in a metric space $X$ to be a condensation point of a set $E \subset X$ if every neighborhood of $p$ contains uncountably many points of $E$.

Suppose $E \subset \mathbb{R}^{k}, E$ is uncountable, and let $P$ be the set of all condensation points of $E$. Prove that $P$ is perfect and that at most countably many points of $E$ are not in $P$. In other words, show that $P^{c} \cap E$ is at most countable.

Proof. We first show $P$ is closed by proving that $P^{c}$ is open. Take a point $p \in P^{c}$. It follows $p$ is not a condensation point so there exists some $r>0$ such that $\left(N_{r}(p) \cap E\right)$ is at most countable. Now we show $N_{r}(p) \subset P^{c}$. Take a point $q \in N_{r}(p)$. Then there exists $\varepsilon$ such that $d(p, q)+\varepsilon<r$. Then $N_{\varepsilon}(q) \subset N_{r}(p)$. Since $N_{r}(p) \cap E$ is at most countable and $N_{\varepsilon}(q) \subset N_{r}(p)$ it follows $N_{\varepsilon}(q) \cap E$ is at most countable. Hence, there exists neighborhood around $q$ whose intersection with $E$ is at most countable. Thus, $q$ is not a condensation point so $q \in P^{c}$. It follows $N_{r}(p) \subset P^{c}$. Hence, $P^{c}$ is open. Then $P$ is closed.

Now we show every point in $P$ is a limit point. For the sake of contradiction, suppose there exists some isolated point $p$. Then there exists some $r>0$ such that $\left(N_{r}(p)-\{p\}\right) \cap P=\varnothing$. Since we have a separable metric space, let $\mathbb{D}$ denote a countable dense subset of $\mathbb{R}^{k}$. Then for all $q \in\left(N_{r}(p)-\{p\}\right) \cap \mathbb{D}$, we have $q$ is not a condensation point. Then for all $q$, take $r_{q}$ as the largest radius such that $N_{r_{q}}(q) \cap E$ is at most countable. This is well-defined since $E$ is uncountable so there is an upper bound to how large the radius will be before it intersects uncountably many points of $E$. Then define the set $T$ as

$$
T=\bigcup_{q \in\left(N_{r}(p)-\{p\}\right) \cap \mathbb{D}} N_{r_{q}}(q)
$$

Now we show $N_{r}(p)-\{p\} \subset T$. By nature of construction of $T$ we know if $x \in\left(N_{r}(p)-\right.$ $\{p\}) \cap \mathbb{D}$ then $x \in T$ so let $x \in\left(N_{r}(p)-\{p\}\right)-\mathbb{D}$. Since $\left(N_{r}(p)-\{p\}\right) \cap P=\varnothing$ we have $x \notin P$ so $x$ is not a condensation point. Then there exists $r_{x}>0$ such that $N_{r_{x}}(x) \cap E$ is at most countable. Then choose $r_{x}$ to be sufficiently small such that $N_{r_{x}}(x) \subset N_{r}(p)$. Moreover, there exists $w \in \mathbb{D}$ such that $d(x, w)<r_{x} / 2$. Then let $\delta=d(x, w)<r_{x} / 2$.

Now we show $N_{\delta}(w) \subset N_{r_{x}}(x)$. Let $a \in N_{\delta}(w)$. Then $d(a, w)<r_{x} / 2$. By the triangle inequality we have

$$
\begin{aligned}
d(a, x) & \leq d(a, w)+d(w, x) \\
& \leq r_{x} / 2+r_{x} / 2 \\
& \leq r_{x}
\end{aligned}
$$

Hence $a \in N_{r_{x}}(x)$. Then $x \in N_{\delta}(w) \subset N_{r_{x}}(x)$. Since $N_{r_{x}}(x) \cap E$ is at most countable, it follows $N_{\delta}(w) \cap E$ is at most countable. Moreover, $w \in N_{r}(p)$ since we chose $r_{x}$ such that $N_{r_{x}}(x) \subset N_{r}(p)-\{p\}$. Then we have $w \in\left(N_{r}(p)-\{p\}\right) \cap \mathbb{D}$ which implies there is an associated neighborhood $N_{r_{w}}(w)$ as one of the sets in the union for $T$. Furthermore, $r_{w}$ is the largest radius such that $N_{r_{w}}(w) \cap E$ is at most countable so then $N_{\delta}(w) \subset N_{r_{w}}(w)$. Since $N_{\delta}(w)$ contains $x$ it follows $x$ is also in $N_{r_{w}}(w)$. Hence $x \in T$. Then we have

$$
\left(N_{r}(p)-\{p\}\right) \cap \mathbb{D} \subset T \text { and }\left(N_{r}(p)-\{p\}\right)-\mathbb{D} \subset T
$$

Hence, $\left(N_{r}(p)-\{p\}\right) \subset T$. Then $T$ is an open cover of $\left(N_{r}(p)-\{p\}\right)$. Now consider

$$
\begin{aligned}
E \cap T & =E \cap \bigcup_{q \in\left(N_{r}(p)-p\right) \cap \mathbb{D}} N_{r_{q}}(q) \\
& =\bigcup_{q \in\left(N_{r}(p)-p\right) \cap \mathbb{D}} E \cap N_{r_{q}}(q)
\end{aligned}
$$

We know $\mathbb{D}$ is countable and that for each $q$ that $E \cap N_{r_{q}}(q)$ is at most countable. It follows that $E \cap T$ is at most a countable union of countable sets which is at most countable. Yet $p$ is a condensation point and we know $\left(N_{r}(p)-\{p\}\right) \cap E$ is uncountable. However,

$$
E \cap\left(N_{r}(p)-\{p\}\right) \subset E \cap T
$$

This is a contradiction. Hence, there does not exist any isolated points. Thus, all points are limit points. Therefore, $P$ is perfect.

Now we show that $P^{c} \cap E$ is at most countable. Note that $\mathbb{R}^{k}$ has a countable base $\mathcal{B}$ that consists of basic open sets $N_{r}(p)$ where $p \in \mathbb{D}$ and $r \in Q$. Moreover, we have $p \in P^{c}$. Since $P$ is closed then $P^{c}$ is open. By definition of a basis, we have $P^{c}=\cup_{\alpha \in J} V_{\alpha}$ where $V_{\alpha} \in \mathcal{B}$ for all $\alpha \in J$ and $J$ is a countable index set. Note that $V_{\alpha} \in P^{c}$ so then $V_{\alpha} \cap E$ is at most countable. Then $E \cap P^{c}=E \cap\left(\cup_{\alpha \in J} V_{\alpha}\right)=\cup_{\alpha \in J}\left(E \cap V_{\alpha}\right)$. It follows we have a countable union of at most countable sets which is at most countable. Hence, $E \cap P^{c}$ is at most countable.

Note that we never used a specific countably dense subset. We just used that fact that one existed and denoted it as $\mathbb{D}$. Moreover, we never used anything in particular to $\mathbb{R}^{k}$. Hence, the argument applies more generally to separable metric spaces.

Exercise (28). Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable. (Corollary: Every countable closed set in $\mathbb{R}^{k}$ has isolated points.)

Proof. Let $C$ be a closed set in a separable metric space. We show that $C=A \cup B$ where $A$ is a perfect set (possibly empty) and $B$ is a set that is at most countable. If $C$ is finite or countable, then let $A=\varnothing$ and $B=C$. Then we are done.

Now suppose $C$ is uncountable. Then let $P$ be the set of all condensation points of $C$. Note that we showed in the previous exercise (27) that $P$ is perfect under separable metric spaces. Moreover, we showed that $P^{c} \cap C$ is at most countable. Hence we have $C=P \cup\left(P^{c} \cap C\right)$. This concludes the proof.

Exercise (29). Prove that every open set in $\mathbb{R}$ is the union of an at most countable collection of disjoint segments.
Proof. Note that $\mathbb{R}$ has a countable dense subset that is $\mathbb{Q}$ so then we have a countable base. Call our countable base $\mathcal{B}$. And an open set $G$ is in $\mathcal{B}$ if and only if $G=N_{r}(p)$ where
$r, p \in \mathbb{Q}$ and $r>0$. By definition of a base, we have every open set in $\mathbb{R}$ is a union of some elements in $\mathcal{B}$. Let $U$ be an open set in $\mathbb{R}$. Then $U=\cup_{V_{n} \in \mathcal{B}^{\prime}} V_{n}$ where $\mathcal{B}^{\prime} \subset \mathcal{B}$. Since $\mathcal{B}$ is countable, it follows $U$ is at most a countable union of open sets. Suppose all open sets $V_{n}$ were disjoint. Then $U$ would be a countable union of disjoint open sets. However, we may have $V_{i} \cap V_{j} \neq \varnothing$ for some $V_{i} \neq V_{j} \in \mathcal{B}^{\prime}$. It follows, $U$ is at most a countable union of disjoint open sets. Hence, every open set in $\mathbb{R}$ is the union of at most a countable collection of disjoint open sets.
Exercise (30). Imitate the proof of Theorem 2.43 to obtain the following result:
If $\mathbb{R}^{k}=\cup_{1}^{\infty} F_{n}$ where $F_{n}$ is a closed subset of $\mathbb{R}^{k}$, then at least one $F_{n}$ has a nonempty interior.

Equivalent statement: If $G_{n}$ is a dense open subset of $\mathbb{R}^{k}$, for $n=1,2,3 \ldots$, then $\cap_{1}^{\infty} G_{n}$ is not empty.
Proof. We prove the equivalent statement first. Let $G_{n}$ be a dense open subset of $\mathbb{R}^{k}$ for all $n$. We construct a sequence of neighborhoods $U_{n}$ recursively as follows.

Let $a \in G_{1}$. Since $G_{1}$ is open, there exists some open neighborhood $N_{r_{a}}(a)$ such that $N_{r_{a}}(a) \subset G_{1}$. Then take $R=r_{a} / 2$. Then let $U_{1}=N_{R}(a)$. It follows $\bar{U}_{1} \subset G_{1}$.

Now suppose we constructed $U_{n}$ such that $\bar{U}_{n} \subset G_{n}$. We now construct $U_{n+1}$ such that $\bar{U}_{n+1} \subset U_{n}$ and $\bar{U}_{n+1} \subset G_{n+1}$. Let $U_{n}=N_{r}(p)$. Since $G_{n+1}$ is dense in $\mathbb{R}^{k}$, there exists $q \in G_{n+1}$ such that $q \in\left(N_{r}(p)-\{p\}\right)$. Moreover, since $q \in G_{n+1}$ and $G_{n+1}$ is open, there exists $\delta>0$ such that $N_{\delta}(q) \subset G_{n+1}$. Then take $\varepsilon$ as

$$
\varepsilon=\min \left\{\frac{\delta}{2}, \frac{r-d(p, q)}{2}\right\}
$$

Then let $U_{n+1}=N_{\varepsilon}(q)$. Now we show $\bar{U}_{n+1} \subset U_{n}$. Let $w \in \bar{U}_{n+1}$. Then $d(q, w) \leq \varepsilon$. Then by the triangle inequality we have

$$
\begin{aligned}
d(p, w) & \leq d(p, q)+d(q, w) \\
& \leq d(p, q)+\frac{r-d(p, q)}{2} \\
& \leq \frac{r+d(p, q)}{2} \\
& <\frac{2 r}{2}=r
\end{aligned}
$$

Thus $d(p, w)<r$ so $w \in N_{r}(p)$. Hence, $\bar{U}_{n+1} \subset U_{n}$.
Now we show $\bar{U}_{n+1} \subset G_{n+1}$. Let $w \in \bar{U}_{n+1}$. Then $d(q, w) \leq \varepsilon \leq \delta / 2<\delta$. Hence, $w \in G_{n+1}$. Thus, $\bar{U}_{n+1} \subset G_{n+1}$. Moreover, since $\bar{U}_{n+1} \subset G_{n+1}$, the recursion can proceed to construct $U_{n+2}$.

Note that $\bar{U}_{n}$ is closed and bounded. Then by Theorem 1.41 it follows $\bar{U}_{n}$ is compact for all $n$. Since we have $\bar{U}_{n} \subset G_{n}$ for all $n$ we have $\cap_{i}^{\infty} \bar{U}_{i} \subset \cap_{i}^{\infty} G_{i}$. Moreover, suppose we have some finite intersection of sets of $\bar{U}_{n}$. Then we have

$$
\bar{U}_{n_{\alpha}}=\bar{U}_{n_{1}} \cap \bar{U}_{n_{2}} \cap \cdots \cap \bar{U}_{n_{m}}
$$

where $n_{\alpha} \geq n_{i}$ for all $1 \leq i \leq m$. This follows because if $n_{\alpha} \geq n_{i}$ for all $i$ then $\bar{U}_{n_{\alpha}} \subset \bar{U}_{n_{i}}$ for all $i$. It follows any finite intersection is non-empty. Moreover, we have $\bar{U}_{1} \supset \bar{U}_{2} \supset \bar{U}_{3} \supset \cdots$ so then by the corollary to Theorem 2.36 we have $\cap_{i=1}^{\infty} \bar{U}_{i} \neq \varnothing$. But we know $\cap_{i}^{\infty} \bar{U}_{i} \subset \cap_{i}^{\infty} G_{i}$ so then $\cap_{i}^{\infty} G_{i}$ must be non-empty.

## 3. Numerical Sequences and Series

Theorem (3.3). Suppose $\left\{s_{n}\right\}$ is a complex sequence and $\lim _{n \rightarrow \infty} s_{n}=s$.
(d) $\lim _{n \rightarrow \infty} \frac{1}{s_{n}}=\frac{1}{s}$, provided that $s_{n} \neq 0$ for all $n \in \mathbb{N}$ and $s \neq 0$.

Proof.
Theorem (3.7). The subsequential limits of a sequence $\left\{p_{n}\right\}$ in a metric space form a closed subset of $X$.

Proof. Let $E$ be the set of all subsequential limits of the sequence $\left\{p_{n}\right\}$. Let $q$ be a limit point of $E$. We show $q$ must be in $E$. If $q$ is a limit point of $E$, then for all $r>0$ we have $\left(N_{r}(q)-\{q\}\right) \cap E \neq \varnothing$. Then take $e_{m} \in\left(N_{1 / m}(q)-\{q\}\right) \cap E$ for each $m \in \mathbb{N}$. Then each $e_{m}$ is a subsequential limit such that $d\left(e_{m}, q\right)<1 / m$.

Now we construct a subsequence of $\left\{p_{n}\right\}$ as follows. Take $n_{1} \in \mathbb{N}$ such that $d\left(e_{1}, p_{n_{1}}\right)<$ $1-d\left(e_{1}, q\right)$. Then by the triangle inequality we have

$$
d\left(q, p_{n_{1}}\right) \leq d\left(q, e_{1}\right)+d\left(e_{1}, p_{n_{1}}\right)<1
$$

Hence, $p_{n_{1}} \in\left(N_{1}(q)-\{q\}\right) \cap P$. More generally, take $n_{k} \in \mathbb{N}$ such that $d\left(e_{k}, p_{n_{k}}\right)<$ $1 / k-d\left(e_{k}, q\right)$ and $n_{k}>n_{k-1}$ which is possible since $e_{k}$ is a subsequential limit of $\left\{p_{n}\right\}$ so there is an infinitely many points in the sequence $\left\{p_{n}\right\}$ that are have a distance less than any $\varepsilon>0$ from $e_{k}$. Again, by the triangle inequality we have

$$
d\left(q, p_{n_{k}}\right) \leq d\left(q, e_{k}\right)+d\left(e_{k}, p_{n_{k}}\right)<1 / k
$$

Hence $p_{n_{k}} \in\left(N_{1 / k}(q)-\{q\}\right) \cap P$. It follows $\left\{p_{n_{k}}\right\}_{k \in \mathbb{N}}$ is a subsequence of $\left\{p_{n}\right\}$.
Now we show $\left\{p_{n_{k}}\right\}_{k \in \mathbb{N}}$ is a subsequence that converges to $q$. Consider $\varepsilon>0$. Then there exists $m \in \mathbb{N}$ such that $0<1 / m<\varepsilon$. By how we constructed $\left\{p_{n_{k}}\right\}_{k \in \mathbb{N}}$, it follows for all

$$
d\left(p_{n_{j}}, q\right)<1 / m<\varepsilon
$$

for all $j \geq m$. Hence, $\lim _{k \rightarrow \infty} p_{n_{k}}=q$. Since, $q$ is a limit of some subsequence of $\left\{p_{n}\right\}$, it follows $q \in E$. Hence, $E$ is closed.

Theorem. If $\left\{p_{n}\right\}$ is a sequence in $X$ and if $E_{N}$ consists of points $p_{N}, p_{N+1}, p_{N+2}, \ldots$, then $\left\{p_{n}\right\}$ is a Cauchy sequence if and only if

$$
\lim _{n \rightarrow \infty} \operatorname{diam} E_{N}=0
$$

Proof. We prove the forward direction first. Let $\left\{p_{n}\right\}$ be a Cauchy sequence. Since $\left\{p_{n}\right\}$ is a Cauchy sequence, then for any $\varepsilon / 2>0$ there exists $M \in \mathbb{N}$ such that $d\left(p_{i}, p_{j}\right)<\varepsilon / 2$ for all $i, j \geq M$. Then let $T=\left\{d\left(p_{i}, p_{j}\right) \mid i, j \geq M\right\}$. It follows $\sup T \leq \varepsilon / 2<\varepsilon$. Note that $\operatorname{diam} E_{M}=\sup T<\varepsilon$. Moreover, since $E_{K} \supset E_{K+1}$ for all $K \in \mathbb{N}$ it follows $\varepsilon>\operatorname{diam} E_{M} \geq \operatorname{diam} E_{M+1} \geq \cdots$. Hence, $\left|\operatorname{diam} E_{N}\right|<\varepsilon$ for all $N \geq M$. Then by definition of converges, we have $\lim _{n \rightarrow \infty} \operatorname{diam} E_{N}=0$.

Now we prove the reverse direction. Suppose $\lim _{n \rightarrow \infty} \operatorname{diam} E_{N}=0$. We show that $\left\{p_{n}\right\}$ must be a Cauchy sequence. Consider $\varepsilon>0$. Since $\lim _{n \rightarrow \infty} \operatorname{diam} E_{N}=0$, there exists $M \in \mathbb{N}$ such that $\left|\operatorname{diam} E_{N}\right|<\varepsilon$ for all $N \geq M$. Then by definition of the diameter of $E_{N}$ we have $d\left(p_{i}, p_{j}\right) \leq \operatorname{diam} E_{M}<\varepsilon$ for all $i, j \geq M$. It follows $\left\{p_{n}\right\}$ is a Cauchy sequence since we found an appropriate $M$ for an arbitrary $e p$.

Theorem (3.10(a)). If $\bar{E}$ is the closure of a set $E$ in a metric space $X$, then

$$
\operatorname{diam} \bar{E}=\operatorname{diam} E
$$

Proof. Since $E \subset \bar{E}$ it follows diam $E \leq \operatorname{diam} \bar{E}$. Moreover, if $E=\bar{E}$ then we are done. Thus, consider the case where $E \subsetneq \bar{E}$. Suppose for the sake of contradiction that diam $E<$ $\operatorname{diam} \bar{E}$. Then take $\alpha$ such that $\operatorname{diam} E<\alpha<\operatorname{diam} \bar{E}$. Since $E \subsetneq \bar{E}$ let $p \in E$ and $q \in \bar{E}-E$ such that $d(p, q)>\alpha$. Note that $q$ is a limit point so then there exists $w \in E$ such that $d(w, q)<d(p, q)-\alpha$. Since $w \in E$ we also have $d(p, w)<\alpha$. Then by the triangle inequality we have

$$
d(p, q) \leq d(p, w)+d(w, q)<d(p, q)
$$

This is a contradiction. Hence, $\operatorname{diam} E \nless \operatorname{diam} \bar{E}$. Therefore, $\operatorname{diam} E=\operatorname{diam} \bar{E}$.
Theorem (3.10(b)). If $K_{n}$ is a sequence of compact sets in $X$ such that $K_{n} \supset K_{n+1}$ for all $n$ and if $\lim _{n \rightarrow \infty}$ diam $K_{n}=0$, then $\cap_{1}^{\infty} K_{n}$ consists of exactly one point.
Proof. Suppose $\cap_{1}^{\infty} K_{n}$ consists of at least two points, say $p$ and $q$. Then $p, q \in K_{n}$ for all $n$. Note that $p \neq q$ so $d(p, q)>0$. Then by definition of diameter we have diam $K_{n} \geq d(p, q)$ for all $n$. However, we must have diam $K_{n}<d(p, q)$ for all but a finite number of $K_{n}$ since $\lim _{n \rightarrow \infty} K_{n}=0$. This is a contradiction. It follows there is at most one point in $\cap_{1}^{\infty} K_{n}$.

Now we show $\cap_{1}^{\infty} K_{n}$ contains exactly one point. Since we have a sequence of nonempty compact sets such that $K_{n} \supset K_{n+1}$ for all $n$, by the corollary to Theorem 2.36, it follows $\cap_{1}^{\infty} K_{n}$ is nonempty. Moreover, we have shown previously that $\cap_{1}^{\infty} K_{n}$ has at most one point. Hence, $\cap_{1}^{\infty} K_{n}$ contains exactly one point.
Theorem (3.11(a)). In any metric space $X$, every convergent sequence is a Cauchy sequence.
Proof. Let $\left\{p_{n}\right\}$ be a convergent sequence in metric space $X$. We show $\left\{p_{n}\right\}$ is a Cauchy sequence. Let $\varepsilon>0$. Since $\left\{p_{n}\right\}$ is a convergent sequence, there exists a point $p \in X$ such that

$$
d\left(p, p_{n}\right)<\varepsilon / 2
$$

for all $n \geq N$ for some $N \in \mathbb{N}$. Then by the triangle inequality we have

$$
d\left(p_{i}, p_{j}\right) \leq d\left(p_{i}, p\right)+d\left(p, p_{j}\right)<\varepsilon / 2+\varepsilon / 2=\varepsilon
$$

for all $i, j \geq N$. Hence $\left\{p_{n}\right\}$ is a Cauchy sequence.
Theorem (3.11(b)). If $X$ is a compact metric space and if $\left\{p_{n}\right\}$ is a Cauchy sequence in $X$, then $\left\{p_{n}\right\}$ converges to some point of $X$.
Proof. Suppose the range of $\left\{p_{n}\right\}$ is finite. Then for point in the range, say $p$, we must have $p=p_{n_{1}}=p_{n_{2}}=\cdots$. Otherwise, if each point in the range has only a finite number of associated points in the sequence, we would must have a finite sequence. However, $\left\{p_{n}\right\}$ is countably infinite. So then $p$ is at least a subsequential limit of $\left\{p_{n}\right\}$. But since $\left\{p_{n}\right\}$ is a Cauchy sequence, it follows that for any $\varepsilon>0$ we have $d\left(p_{i}, p_{j}\right)<\varepsilon$ for all $i, j \geq N$ for some $N \in \mathbb{N}$. In particular, we must be able to get arbitrarily close to the at least subsequential limit $p$. Hence, for any $\varepsilon>0$ we have $d\left(p, p_{i}\right)<\varepsilon$ for all $i \geq N$ for some $N \in \mathbb{N}$. It follows that $\left\{p_{n}\right\}$ must converge to point $p$.

Now suppose the range of $\left\{p_{n}\right\}$ is infinite. Note that $X$ is a compact metric space which implies any infinite subset of $X$ has a limit point in $X$. Since the range of $\left\{p_{n}\right\}$ is infinite, it follows $\left\{p_{n}\right\}$ has a limit point in $X$, call $p$. Let $\varepsilon>0$. Since $p$ is a limit point, the neighborhood $N_{\varepsilon}(p)$ contains infinitely many points of $\left\{p_{n}\right\}$.

For the sake of contradiction, suppose $N_{\varepsilon}(p)$ does not contain infinitely many points of $\left\{p_{n}\right\}$. Then for all $N \in \mathbb{N}$, there exists $i \geq N$ such that $d\left(p, p_{i}\right) \geq \varepsilon$. Moreover, since $\left\{p_{n}\right\}$ is a Cauchy sequence, for $\varepsilon / 2$ there exists $M \in \mathbb{N}$ such that $d(p) \ldots$
Theorem (3.11(c)). In $\mathbb{R}^{k}$, every Cauchy sequence converges.
Proof.

## 4. Continuity

Theorem (4.2). Let $X, Y, E, f$, and $p$ be as in Definition 4.1. Then

$$
\lim _{x \rightarrow p} f(x)=q
$$

if and only if

$$
\lim _{n \rightarrow \infty} f\left(p_{n}\right)=q
$$

for every sequence $\left(p_{n}\right)$ in $E$ such that

$$
p_{n} \neq p \text { and } \lim _{n \rightarrow \infty} p_{n}=p
$$

Proof. Let $X, Y, E, f$, and $p$ be defined as in Definition 4.1. We prove the forward direction first. Let $\lim _{x \rightarrow p} f(x)=q$ and suppose we have a sequence $\left(p_{n}\right)_{n \in \mathbb{N}}$ in $E$ such that $p_{n} \neq p$ and $\lim _{n \rightarrow \infty} p_{n}=p$. We show that $\lim _{n \rightarrow \infty} f\left(p_{n}\right)=q$. It suffices to show that for $\varepsilon>0$ there exists $N \in \mathbb{N}$ such that $d_{Y}\left(f\left(p_{n}\right), q\right)<\varepsilon$ for all $n \geq N$. Note that $\lim _{x \rightarrow p} f(x)=q$ so then there exists $\delta>0$ such that $d_{Y}(f(x), q)<\varepsilon$ for all $x$ where $0<d_{X}(x, p)<\delta$. Moreover, since $\lim _{n \rightarrow \infty} p_{n}=p$ and $p_{n} \neq p$ it follows there exists $N$ such that $0<d_{X}\left(p_{n}, p\right)<\delta$ for all $n \geq N$. Hence, $d_{Y}\left(f\left(p_{n}\right), q\right)<\varepsilon$ for all $n \geq N$. Then by definition of a limit of a sequence, we have $\lim _{n \rightarrow \infty} f\left(p_{n}\right)=q$.

Now we prove the reverse direction. We prove the contrapositive. Suppose $\lim _{x \rightarrow p} f(x) \neq$ $q$. Then for some $\varepsilon>0$, for all $\delta>0$, there exists $x \in X$ such that $0<d_{X}(x, p)<\delta$ yet $d_{Y}(f(x), q) \geq \varepsilon$. Then we construct the sequence $\left(p_{n}\right)_{n \in \mathbb{N}}$ as follows. For all $n \in \mathbb{N}$, take $p_{n}$ such that $d_{X}\left(p_{n}, p\right)<1 / n$ and $p_{n} \neq p$ and $d_{Y}\left(f\left(p_{n}\right), q\right) \geq \varepsilon$. We show $\lim _{n \rightarrow \infty} p_{n}=p$. Consider $\varepsilon^{\prime}>0$. Then there exists a natural number $m$ such that $1 / m<\varepsilon^{\prime}$. Then $d_{X}\left(p_{n}, p\right)<$ $1 / m<\varepsilon^{\prime}$ for all $n \geq m$. Hence, $p_{n} \rightarrow p$. Now we show $\lim _{n \rightarrow \infty} f\left(p_{n}\right) \neq q$. By nature of construction of $\left(p_{n}\right)$ we have $d_{Y}\left(f\left(p_{n}\right), q\right) \geq \varepsilon$ for all $n \in \mathbb{N}$. Hence, $\lim _{n \rightarrow \infty} f\left(p_{n}\right) \neq q$. This proves the contrapositive.
Theorem (4.4). Suppose $E \subset X$, a metric space, $p$ is a limit point of $E, f$ and $g$ are complex functions on $E$, and

$$
\lim _{x \rightarrow p} f(x)=A \text { and } \lim _{x \rightarrow p} g(x)=B
$$

Then
(a) $\lim _{x \rightarrow p}(f+g)(x)=A+B$
(b) $\lim _{x \rightarrow p}(f g)(x)=A B$
(c) $\lim _{x \rightarrow p}(f / g)(x)=A / B$, if $B \neq 0$.

Proof. Suppose $E \subset X$, a metric space, $p$ is a limit point of $E$, and $f$ and $g$ are complex functions on $E$, and $\lim _{x \rightarrow p} f(x)=A$ and $\lim _{x \rightarrow p} g(x)=B$. Then by Theorem 4.2 we have for every sequence $\left(p_{n}\right)$ where $p_{n} \neq p$ and $p_{n} \rightarrow p$ that $\lim _{n \rightarrow \infty} f\left(p_{n}\right)=A$ and $\lim _{n \rightarrow \infty} g\left(p_{n}\right)=B$. By Theorem 3.3 we know

$$
\lim _{n \rightarrow \infty} f\left(p_{n}\right)+g\left(p_{n}\right)=\lim _{n \rightarrow \infty}(f+g)\left(p_{n}\right)=A+B
$$

for all sequences $\left(p_{n}\right)$. Hence, $\lim _{x \rightarrow p}(f+g)(x)=A+B$. Similar arguments prove parts (b) and (c).

Theorem (4.6). In the situation given in Definition 4.5, assume also that $p$ is a limit point of $E$. Then $f$ is continuous at $p$ if and only if $\lim _{x \rightarrow p} f(x)=f(p)$.
Proof. Let $p$ be a limit point of $E$. We prove the forward direction first. Suppose $f$ is continuous at $p$. Then for every $\varepsilon$ there exists $\delta$ such that for all $x \in E$ where $d_{X}(x, p)<\delta$ we have $d_{Y}(f(x), f(p))<\varepsilon$. Let $A$ be the set defined as

$$
A=\left\{x \in E \mid d_{X}(x, p)<\delta\right\}
$$

Moreover, since $p$ is a limit point of $E$ it follows $(A-\{p\}) \cap E \neq \varnothing$. Define $B$ as the set

$$
B=\left\{x \in E \mid 0<d_{X}(x, p)<\delta\right\}
$$

Then $B$ is nonempty. Moreover, since $B \subset A$ it follows if $0<d_{X}(x, p)<\delta$ and $x \in E$ then $d_{Y}(f(x), f(p))<\varepsilon$. Hence, $\lim _{x \rightarrow p} f(x)=f(p)$.

Now we prove the reverse direction. Suppose $\lim _{x \rightarrow p} f(x)=f(p)$. Let $\varepsilon>0$. Then there exists $\delta>0$ such that if $0<d_{X}(x, p)<\delta$ and $x \in E$ then $d_{Y}(f(x), f(p))<\varepsilon$. Furthermore, if $d_{X}(x, p)=0$ then $x=p$ and $d_{Y}(f(p), f(p))=0<\varepsilon$. Hence if $d_{X}(x, p)<\delta$ and $x \in E$ then $d_{Y}(f(x), f(p))<\varepsilon$. It follows $f$ is continuous at $p$.
Theorem (4.7). Suppose $X, Y, Z$ are metric spaces, $E \subset X, f$ maps $E$ into $Y, g$ maps the range of $f, f(E)$, into $Z$, and $h$ is the mapping of $E$ into $Z$ defined by

$$
h(x)=g(f(x)), x \in E
$$

If $f$ is continuous at a point $p \in E$ and if $g$ is continuous at the point $f(p)$, then $h$ is continuous at $p$.
Proof. Take $\varepsilon>0$. Since $g$ is continuous at the point $f(p)$, there exists $\eta>0$ such that if $y \in f(E)$ and $d_{Y}(y, f(p))<\eta$ then $d_{Z}(g(y), g(f(p)))<\varepsilon$. Moreover, since $f$ is continuous at point $p$, it follows there exists $\delta>0$ such that if $x \in X$ and $d_{X}(x, p)<\delta$ then $d_{Y}(f(x), f(p))<\eta$. But this implies $d_{Z}(g(f(x)), g(f(p)))<\varepsilon$ or $d_{Z}(h(x), h(p))<\varepsilon$. It follows $h$ is continuous at point $p$.

Theorem (4.8). A mapping $f$ of a metric space $X$ into a metric space $Y$ is continuous on $X$ if and only if $f^{-1}(V)$ is open in $X$ for every open set $V$ in $Y$.
Proof. We prove the forward direction first.
Note. If $x$ is a point in the domain of definition of the function $f$ at which $f$ is not continuous, then $f$ is discontinuous at $x$. Note that if we say $f$ is continuous at a point $p$ then it follows $p$ is within the domain definition of $f$.
Definition (4.25). Let $f$ be defined on $(a, b)$. Consider any point $x$ such that $a \leq x<b$. We write

$$
f(x+)=q
$$

if $f\left(t_{n}\right) \rightarrow q$ as $n \rightarrow \infty$, for all sequences $\left(t_{n}\right)_{n \in \mathbb{N}}$ in $(x, b)$ such that $t_{n} \rightarrow x$. To obtain the definition of $f(x-)$, for $a<x \leq b$, we restrict ourselves to sequences $\left(t_{n}\right)$ in $(a, x)$.
Remark. Note that the above definitions for left-handed and right-handed limits are generalizations to the definition of a limit in Theorem 4.2. In fact, this definition uses the same idea as Theorem 4.2. Note that for $f(x+)$ we allow for the case $x=a$ even though $f$ is only defined on $(a, b)$. This is an important observation about limits, the limit of a map $f$ at a particular point $x$ for may be defined even though $f$ may not be defined at $x$. Consider the
case where $f:(-1,1) \rightarrow \mathbb{R}$ defined as $f(x)=x^{2}$. It is intuitive there is a right hand limit at $x=-1$ and a left hand limit at $x=1$.

Lemma. Let $f$ be defined on $(a, b)$ and $x \in(a, b)$. Then $\lim _{t \rightarrow x} f(t)$ exists if and only if

$$
f(x+)=f(x-)=\lim _{t \rightarrow x} f(t)
$$

Definition (4.26). Let $f$ be defined on $(a, b)$. If $f$ is discontinuous at a point $x$ and if $f(x+)$ and $f(x-)$ exist, then $f$ is said to have a discontinuity of the first kind, or a simply discontinuity, at $x$. Otherwise the discontinuity is said to be of the second kind.
Example. Let $f: \mathbb{R} \rightarrow \mathbb{R}$. Then define $f(x)$ as

$$
f(x)= \begin{cases}x & x \neq 0 \\ 1 & x=0\end{cases}
$$

This has a simple discontinuity at $x=0$ since left hand and right hand limits are 0 . Consider $g: \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$
g(x)= \begin{cases}-1 & x<0 \\ 1 & x \geq 0\end{cases}
$$

Then the left hand limit at $f(0+)=-1$ and $f(0-)=1$. Hence, $f$ has a discontinuity of the second kind at $x=0$.

Example (4.27). Three examples from the book.
(a) Define

$$
f(x)= \begin{cases}1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R}-\mathbb{Q}\end{cases}
$$

Then $f$ has a discontinuity of the second kind at every point $x$, since neither $f(x+)$ nor $f(x-)$ exists.
(b) Define

$$
f(x)= \begin{cases}x & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R}-\mathbb{Q}\end{cases}
$$

Then $f$ is continuous at $x=0$ and has a discontinuity of the second kind at every other point.
(c)

Definition (4.28). Let $f$ be real on $(a, b)$. Then $f$ is said to be monotonically increasing on $(a, b)$ if $a<x<y<b$ implies $f(x) \leq f(y)$. If the last inequality is reversed, we obtain the definition of a monotonically decreasing function. The class of monotonic functions consist of both the increasing and decreasing functions.
Theorem (4.29). Let $f$ be monotonically increasing on $(a, b)$. Then $f(x+)$ and $f(x-)$ exist at every point of $x$ of $(a, b)$. More precisely,

$$
\sup _{a<t<x} f(t)=f(x-) \leq f(x) \leq f(x+)=\inf _{x<t<b} f(t)
$$

Furthermore, if $a<x<y<b$, then

$$
f(x+) \leq f(y-)
$$

Analogous results hold for monotonically decreasing functions.

Proof. We first prove that $f(x-)$ exists at every point $x \in(a, b)$. Arbitrarily choose $x \in(a, b)$. Define the set $A$ as $A=\{f(t) \in \mathbb{R} \mid a<t<x\}$. Since $f$ is monotonically increasing, it follows $f(t) \leq f(x)$ for all $a<t<x$. Hence, $f(x)$ is an upper bound of $A$. Moreover, $f$ is a real valued function so $A \subset \mathbb{R}$ and $A$ is non-empty since there exists a real number in the open interval $(a, x)$ and $f$ is defined for $(a, b) \supset(a, x)$. Hence, $A$ has a least upper bound. Take $\alpha=\sup A$.

For the sake of contradiction suppose there exists a sequence $\left(t_{n}\right)_{n \in \mathbb{N}}$ in $(a, x)$ such that $\lim _{n \rightarrow \infty} t_{n}=x$ but $\lim _{n \rightarrow \infty} f\left(t_{n}\right) \neq \alpha$. Then for some $\varepsilon>0$ for all $M \in \mathbb{N}$ there exists $m \geq M$ such $\left|\alpha-f\left(t_{m}\right)\right| \geq \varepsilon$. Moreover, $\alpha>f\left(t_{m}\right)$ by definition of $\alpha$ so then equivalently we have $\alpha-f\left(t_{m}\right) \geq \varepsilon$.

Now take $c \in(a, x)$. Then $a<c<x$. Since $\lim _{n \rightarrow \infty} t_{n}=x$ there exists $I \in \mathbb{N}$ such that $\left|x-t_{i}\right|<x-c$ for all $i \geq I$. Since $t_{i} \in(a, x), t_{i}<x$. Then $x-t_{i}<x-c$ for all $i \geq I$. Then $c<t_{i}$ for all $i \geq I$. By the previous paragraph, there exists $m_{0} \geq I$ such that $\alpha-f\left(t_{m_{0}}\right) \geq \varepsilon$. Equivalently, $\alpha-\varepsilon \geq f\left(t_{m_{0}}\right)$. Since $m_{0} \geq I$ it follows $t_{m_{0}}>c$. Note $f$ is monotonically increasing so $f(c) \leq f\left(t_{m_{0}}\right) \leq \alpha-\varepsilon$. Our choice of $c$ was arbitrary. Hence $f(c) \leq \alpha-\varepsilon$ for all $c \in(a, b)$. It follows $\alpha-\varepsilon$ is an upper bound of $A$. This contradicts $\alpha$ being the least upper bound of $A$. Therefore, the sequence $\left(t_{n}\right)_{n \in \mathbb{N}}$ does not exist. Hence, for every sequence $\left(s_{n}\right)_{n \in \mathbb{N}}$ in $(a, x)$ where $\lim _{n \rightarrow \infty} s_{n}=x$ we must have $\lim _{n \rightarrow \infty} f\left(s_{n}\right)=\alpha$. It follows $f(x-)=\alpha$. We also proved in the first paragraph that $f(x)$ is an upper bound of $A$. In total we have

$$
\sup _{a<t<x} f(t)=f(x-) \leq f(x)
$$

for all $x \in(a, b)$. An analogous argument shows

$$
f(x) \leq f(x+)=\inf _{x<t<b} f(t)
$$

Now we show that if $a<x<y<b$, then $f(x+) \leq f(y-)$. We prove by contradiction. Suppose $f(x+)>f(y-)$. Then let $\left(x_{n}\right)_{n \in \mathbb{N}}$ be a sequence in $(x, b)$ such that $\lim _{n \rightarrow \infty} x_{n}=x$ and $\lim _{n \rightarrow \infty} f\left(x_{n}\right)=f(x+)$ and let $\left(y_{n}\right)_{n \in \mathbb{N}}$ be a sequence in $(a, y)$ such that $\lim _{n \rightarrow \infty} y_{n}=y$ and $\lim _{n \rightarrow \infty} f\left(y_{n}\right)=f(y+)$. Then take $\varepsilon_{1}=(f(x+)-f(y-)) / 2$ and $\varepsilon_{2}=(y-x) / 2$. Then there exists $N_{1}$ such that $\left|f(x+)-f\left(x_{n}\right)\right|<\varepsilon_{1}$ for all $n \geq N_{1}$ and $N_{2}$ such that $\left|x-x_{n}\right|<\varepsilon_{2}$ for all $n \geq N_{2}$. Similarly, there exists $M_{1}$ such that $\left|f(y-)-f\left(y_{m}\right)\right|<\varepsilon_{1}$ for all $m \geq M_{1}$ and $M_{2}$ such that $\left|y-y_{m}\right|<\varepsilon_{2}$ for all $m \geq M_{2}$. Take $N=\max \left\{N_{1}, N_{2}\right\}$ and $M=\max \left\{M_{1}, M_{2}\right\}$. Now we use these inequalities to show a contradiction.

By the four inequalities we have

$$
\begin{gathered}
\frac{f(x+)+f(y-)}{2}=f(x+)-\varepsilon_{1}<f\left(x_{n}\right)<f(x+)+\varepsilon_{1} \\
f(y-)-\varepsilon_{1}<f\left(y_{m}\right)<f(y-)+\varepsilon_{1}=\frac{f(x+)+f(y-)}{2} \\
x-\varepsilon_{2}<x_{n}<x+\varepsilon_{2}=\frac{x+y}{2} \\
\frac{x+y}{2}=y-\varepsilon / 2<y_{m}<y+\varepsilon / 2
\end{gathered}
$$

for all $n \geq N$ and $m \geq M$. Then $x_{n}<y_{m}$ but $f\left(x_{n}\right)>f\left(y_{m}\right)$ for all $n \geq N$ and $m \geq M$. Take $i \geq N$ and $j \geq M$. Then we have $x_{i}<y_{j}$ but $f\left(x_{i}\right)>f\left(y_{j}\right)$ which is a contradiction to $f$ being monotonically increasing. It follows $f(x+) \leq f(y-)$.

Exercise (1). Suppose $f$ is a real function defined on $\mathbb{R}$ which satisfies

$$
\lim _{h \rightarrow 0}[f(x+h)-f(x-h)]=0
$$

for every $x \in \mathbb{R}$. Does this imply $f$ is continuous?
Proof. This does not imply $f$ is continuous. Define $f: \mathbb{R} \rightarrow \mathbb{R}$ as

$$
f(x)= \begin{cases}1 & x \neq 0 \\ 0 & x=0\end{cases}
$$

First consider when $x>0$. Then let $\varepsilon>0$. Then take $\delta$ such that $0<\delta<x$. Then if $0<|h-0|<\delta$ we have $x+h, x-h>0$. It follows $|(f(x+h)-f(x-h))-0|=|1-1-0|=0<\varepsilon$. Hence, the limit holds when $x>0$. A similar argument shows the limit holds when $x<0$. Now consider when $x=0$. Let $\varepsilon>0$. Then let $\delta>0$ be arbitrary. Then if $h \in \mathbb{R}$ and $0<|h-0|<\delta$ then $x-h=-h$ and $x+h=h$. Then either $-h>0$ and $h<0$ or $-h<0$ and $h>0$. For both cases $f(x+h)=f(x-h)$. It follows $|(f(x+h)-f(x-h))-0|=0<\varepsilon$. Hence, the limit holds for all $x$. However, $f$ is not continuous at $x=0$. In fact, it is a discontinuity of the first kind.

Exercise (2). If $f$ is a continuous mapping of a metric space $X$ into a metric space $Y$, prove that

$$
f(\bar{E}) \subset \overline{f(E)}
$$

for every set $E \subset X$. Show, by an example, that $f(\bar{E})$ can be a proper subset of $\overline{f(E)}$.
Proof. Let $f$ be a continuous mapping of a metric space $X$ into a metric space $Y$. We show $f(\bar{E}) \subset \overline{f(E)}$ for all $E \subset X$. Let $E \subset X$ and $y \in f(\bar{E})$. Then there exists $x \in \bar{E}$ such that $f(x)=y$. There are two cases. First suppose $x \in E$. Then $y=f(x) \in f(E) \subset \overline{f(E)}$. Secondly, consider the case where $x \in E^{\prime}$ where $E^{\prime}$ denotes the set of all limits points of $E$. We show either $f(x) \in f(E)$ or $f(x)$ is a limit point of $f(E)$. Take $\varepsilon>0$. Since $f$ is continuous at $x$ for $\varepsilon>0$ there exists $\delta>0$ such that if $p \in E$ and $d_{X}(p, x)<\delta$ then $d_{Y}(f(p), f(x))<\varepsilon$. Since $x$ is a limit point of $E$ we know $\left(N_{\delta}^{X}(x)-\{x\}\right) \cap E \neq \varnothing^{1}$. Now suppose $f\left(\left(N_{\delta}^{X}(x)-\{x\}\right) \cap E\right)=\{f(x)\}$. Since $f\left(\left(N_{\delta}^{X}(x)-\{x\}\right) \cap E\right) \subset f(E)$ it follows $f(x) \in f(E)$. Then $y \in \overline{f(E)}$.

Now suppose $f\left(\left(N_{\delta}^{X}(x)-\{x\}\right) \cap E\right) \neq\{f(x)\}$. Note that $f\left(N_{\delta}^{X}(x) \cap E\right) \subset N_{\varepsilon}^{Y}(f(x))$ and then $f\left(\left(N_{\delta}^{X}(x)-\{x\}\right) \cap E\right) \subset N_{\varepsilon}^{Y}(f(x))$. Since $f\left(\left(N_{\delta}^{X}(x)-\{x\}\right) \cap E\right) \neq\{f(x)\}$ it follows

$$
\varnothing \neq f\left(\left(N_{\delta}^{X}(x)-\{x\}\right) \cap E\right)-\{f(x)\} \subset N_{\varepsilon}^{Y}((f(x))-\{f(x)\}
$$

It follows $\left(N_{\varepsilon}^{Y}((f(x))-\{f(x)\}) \cap f(E) \neq \varnothing\right.$ for all $\varepsilon>0$. Hence, $f(x)=y$ is a limit point of $f(E)$. Then $y \in \overline{f(E)}$.

An example where $f(\bar{E}) \subsetneq \overline{f(E)}$ is the following. Let $X=\mathbb{Q}$ and $Y=\mathbb{R}$. Then define $f: X \rightarrow Y$ as $f(x)=x$. Moreover, define the distance functions as $d_{X}: \mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{R}$ as $d_{X}(p, q)=|p-q|$ and $d_{Y}: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ as $d_{Y}(p, q)=|p-q|$. Now we show $f$ is continuous on $\mathbb{Q}$. Let $p \in \mathbb{Q}$. Let $\varepsilon>0$. Then let $\delta=\varepsilon$. It follows if $d_{X}(p, q)<\delta$ and $q \in \mathbb{Q}$ then $d_{Y}(f(p), f(q))=d_{Y}(p, q)<\delta=\varepsilon$. Hence, $f$ is continuous on $\mathbb{Q}$. Let $E=\mathbb{Q}=X$. Then $\bar{E}=X$. Hence, $f(\bar{E})=\mathbb{Q}$. And $f(E)=\mathbb{Q}$ and $\overline{f(E)}=\overline{\mathbb{Q}}=\mathbb{R}$. Therefore, $f(\bar{E}) \subsetneq \overline{f(E)}$.
${ }^{1} N_{\delta}^{X}(x)$ denotes the neighborhood of radius $\delta$ about point $x$ with the distance function $d_{X}$ associated with metric space $\left(X, d_{X}\right)$.

Exercise (3). Let $f$ be a continuous real function on a metric space $X$. Let $Z(f)$ be the set of all $p \in X$ at which $f(p)=0$. Prove that $Z(p)$ is closed.

Proof. Note $f$ is a real function so $f(X) \subset \mathbb{R}$. We show $X-Z(f)$ is open. Let $q \in X-Z(f)$. Then $f(q) \neq 0$. Take $\varepsilon=|f(q)|$. Since $f$ is continuous there exists $\delta$ such that for all $x \in X$ and $d_{X}(x, q)<\delta$ we have $d_{Y}(f(x), f(q))<\varepsilon=|f(q)|$. Now let $w \in N_{\delta}^{X}(q)$. Then $d_{X}(w, q)<\delta$ so $d_{Y}(f(w), f(q))<\varepsilon$. Then

$$
|f(w)-f(q)|<|f(q)|
$$

Then

$$
-|f(q)|<f(w)-f(q)<|f(q)|
$$

Then

$$
f(q)-|f(q)|<f(w)<f(q)+|f(q)|
$$

Then either $f(w)>0$ or $f(w)<0$ so in either case $f(w) \neq 0$. It follows $N_{\delta}^{X}(q) \cap Z(f)=\varnothing$. So $N_{\delta}^{X}(q) \subset X-Z(f)$. Hence, $X-Z(f)$ is open and $Z(f)$ is closed.
Exercise (4). Let $f$ and $g$ be continuous mappings of a metric space $X$ into a metric space $Y$ and let $E$ be a dense subset of $X$. Prove that $f(E)$ is dense in $f(X)$. If $g(p)=f(p)$ for all $p \in E$, prove that $g(p)=f(p)$ for all $p \in X$. (In other words, a continuous mapping is determined by its values on a dense subset of its domain.)
Proof. First we show that if $E$ is a dense subset of $X$, then $f(E)$ is dense in $f(X)$. To show $f(E)$ is dense in $f(X)$ it suffices to show that every point in $f(X)$ is in $f(E)$ or is a limit point of $f(E)$. Let $y \in f(X)$. Then there exists $x \in X$ such that $f(x)=y$. There are two cases: either there exists $x \in E$ such that $f(x)=y$ or there does not exist $x \in E$ such that $f(x)=y$. If there exists $x \in E$ such that $f(x)=y$ then $y \in f(E)$. Now suppose there does not exist $x \in E$ such that $f(x)=y$. Then we have $f\left(x_{0}\right)=y$ for $x_{0} \in E^{c}$. Take $\varepsilon>0$. Then consider the open neighborhood $N_{\varepsilon}^{Y}(y)$ in $Y$. Since $f$ is continuous it follows $f^{-1}\left(N_{\varepsilon}^{Y}(y)\right)$ is open in $X$. Moreover, $x_{0} \in f^{-1}\left(N_{\varepsilon}^{Y}(y)\right)$. Since $f^{-1}\left(N_{\varepsilon}^{Y}(y)\right)$ and $x_{0} \in f^{-1}\left(N_{\varepsilon}^{Y}(y)\right)$, there exists $\delta>0$ such that $N_{\delta}^{X}\left(x_{0}\right) \subset f^{-1}\left(N_{\varepsilon}^{Y}(y)\right)$. Moreover, because $E$ is dense in $X$ and $x_{0} \notin E$ it follows $\left(N_{\delta}^{X}\left(x_{0}\right)-\left\{x_{0}\right\}\right) \cap E \neq \varnothing$. Let $q \in\left(N_{\delta}^{X}(x)-\{x\}\right) \cap E$. Then $f\left(N_{\delta}^{X}\left(x_{0}\right)\right) \subset N_{\varepsilon}^{Y}(y)$ and $f(q) \in N_{\varepsilon}^{Y}(y)$ but also $f(q) \in f(E)$. Since $y \notin f(E)$ it follows $f(q) \neq y$. It follows $f(q) \in\left(N_{\varepsilon}^{Y}(y)-\{y\}\right) \cap f(E)$ so $\left.\left(N_{\varepsilon}^{Y}(y)-\{y\}\right) \cap f(E)\right) \neq \varnothing$. Hence, $y$ is a limit point of $f(E)$. It follows $f(E)$ is dense in $f(X)$.

Now we show if $g(p)=f(p)$ for all $p \in E$, then $g(p)=f(p)$ for all $p \in X$. Let $q \in X-E$. For the sake of contradiction, suppose $g(q) \neq f(q)$. Then $d_{Y}(g(q), f(q)) \neq 0$. Then take $\delta=\frac{1}{2} d_{Y}(g(q), f(q))$. Then $N_{\delta}^{Y}(g(q)) \cap N_{\delta}^{Y}(f(q))=\varnothing$. Since $g$ and $f$ are continuous mappings it follows their inverse images are open. Moreover, $q \in g^{-1}\left(N_{\delta}^{Y}(g(q))\right) \cap f^{-1}\left(N_{\delta}^{Y}(f(q))\right)$. This is a finite intersection of open sets which is open. Then consider some open neighborhood around $q$, say $V_{q}$, in $X$ which is a subset of this intersection, that is $V_{q} \subset g^{-1}\left(N_{\delta}^{Y}(g(q))\right) \cap$ $f^{-1}\left(N_{\delta}^{Y}(f(q))\right)$. Since $E$ is dense in $X$ it follows $E \cap V_{q} \neq \varnothing$. Moreover, $q \notin E$ so then there exists $w \neq q$ such that $w \in E \cap V_{q}$. Since $w \in E$ it follows $f(w)=g(w)$. Then $f(w)=g(w) \in$ $N_{\delta}^{Y}(g(q))$ and $f(w)=g(w) \in N_{\delta}^{Y}(f(q))$. This contradicts the fact $N_{\delta}^{Y}(g(q)) \cap N_{\delta}^{Y}(f(q))=\varnothing$. Hence, $g(q)=f(q)$ for all $q \in X-E$. By the hypothesis we know $g(p)=f(p)$ for all $p \in E$. Hence, $g(x)=f(x)$ for all $x \in X$.

Exercise (5).
Proof.

Exercise (6). If $f$ is defined on $E$, the graph of $f$ is the set of points $(x, f(x))$, for $x \in E$. In particular, if $E$ is the set of real numbers, and $f$ is real-valued, the graph of $f$ is a subset of the plane. Suppose $E$ is compact, and prove that $f$ is continuous on $E$ if and only if its graph is compact.

Proof. Let $G=\left\{(x, f(x)) \in \mathbb{R}^{2} \mid x \in E\right\}$. We prove the forward direction first. We know $E$ is compact and $f$ is continuous on $E$. Then $f(E)$ is compact and $f$ is uniformly continuous on $E$. To show $G$ is compact, it is sufficient to show that every infinite subset of $G$ has a limit point in $G$. Let $U$ be an infinite subset of $G$. Define the projection function $\pi: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ as $\pi(a, b)=a$ for all $a, b \in \mathbb{R}$. Then let $U_{x}=\pi(U)$. Let $\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$ be distinct points of $U$. Since $U \subset G$ and $G$ is the graph of the function $f$ it follows $x_{1} \neq x_{2}$. Hence, $U_{x}$ has at least the same cardinality as $U$. Then $U_{x}$ is an infinite subset of $E$. Since $E$ is compact, it follows $U_{x}$ has a limit point in $E$, say $x_{0}$.

Now we show $\left(x_{0}, f\left(x_{0}\right)\right) \in G$ is a limit point of $U$. Since $x_{0}$ is a limit point of $U_{x}$, there exists a sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ such that $x_{n} \neq x$ for all $n$ and $\lim _{n \rightarrow \infty} x_{n}=x$. Moreover, since $f$ is continuous and $x_{0}$ is a limit point we have $\lim _{n \rightarrow \infty} f\left(x_{n}\right)=f\left(x_{0}\right)$. Take $\varepsilon>0$. Then there exists $\delta>0$ such that if $p \in E$ and $0<\left|p-x_{0}\right|<\delta$ then $\left|f(p)-f\left(x_{0}\right)\right|<\varepsilon / 2$. Moreover, we can choose $\delta$ to be sufficiently small such that $\sqrt{\delta^{2}+(\varepsilon / 2)^{2}}<\varepsilon$. Furthermore, since $x_{n} \rightarrow x$ it follows there exist $x_{i} \in U_{x}$ such that $0<\left|x_{i}-x_{0}\right|<\delta$ so then $\left|f\left(x_{i}\right)-f\left(x_{0}\right)\right|<\varepsilon / 2$. Then we have

$$
0<\sqrt{\left(x_{i}-x_{0}\right)^{2}+\left(f\left(x_{i}\right)-f\left(x_{0}\right)\right)^{2}}<\varepsilon
$$

so then

$$
\left(x_{i}, f\left(x_{i}\right)\right) \in\left(N_{\varepsilon}\left(x_{0}, f\left(x_{0}\right)\right)-\left\{\left(x_{0}, f\left(x_{0}\right)\right)\right\}\right) \cap U
$$

It follows $\left(x_{0}, f\left(x_{0}\right)\right)$ is a limit point of $U$. Then every infinite subset of $G$ has a limit point in $G$. Hence, $G$ is compact.

Now we prove the reverse direction.
Exercise (9). Show that the requirement in the definition of uniform continuity can be rephrased as follows, in terms of diameters of sets: To every $\varepsilon>0$ there exists a $\delta>0$ such that $\operatorname{diam} f(E)<\varepsilon$ for all $E \subset X$ with $\operatorname{diam} E<\delta$.

